NUMERICAL LUNAR THEORY

BY

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LATE ASTRONOMER ROYAL

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PREFACE

THE treatise which I now place before the public is not offered as bearing the character of a Complete Theory of the motions of the Moon. It is rather an Examination of Lunar Theory, as tested by the substitution of numbers, for symbols, or for the results of long and complicated operations conducted exclusively by use of symbols; and, for this reason, the distinctive word "Numerical" is adopted in its title. It was begun almost accidentally, from an inspection of the differential equations of the Moon's motions, and a trial how far these equations would be satisfied by numerical values of co-efficients of a limited number of terms furnished by Delaunay's theory. To my great surprise, large discordances appeared. An opportunity presented itself, of communicating these results to the Board of Visitors of the Royal Observatory, at one of their periodical meetings. On the representation made by that body to the Board of Admiralty, and through them to the Lords Commissioners of Her Majesty's Treasury, authority was given to me to proceed with the investigation, and to the Royal Stationery Office to print the work, when completed, at the public expense.

It was understood that all calculations would be carried to the 7th decimal of the adopted unit (the Moon's mean distance), corresponding nearly to $\frac{1}{50}$ of a second of arc. Calculations were made in duplicate, and the mass became heavy and expensive. During my connexion with the Royal Observatory, sums were allowed on the Observatory Estimates, for the expense of calculations. After my resignation of the Office of Astronomer Royal (and indeed before the last estimate could be drawn), I had no further public assistance. For reimbursement of expenses which I had incurred, and for future expenses, a pecuniary contribution was made by a Member of the Board of Visitors, well known for his scientific enterprise in a totally different direction, but who had also proved, by extensive private outlay, and by very successful use of a special class of instruments, the interest which he took in astronomical inquiries.

The work was greatly delayed by the heavy pressure of business, not only in the ordinary conduct of the Observatory, but also in completing the preparations, reports, and calculations, for the Transit of Venus of the year 1874, and in preparing for that of 1882.

On the work itself, I now offer some remarks. I have explained above that the principle of operations was, to arrange the fundamental mechanical equations in a form suited for the investigations of Lunar Theory; to substitute in the terms of these equations the numerical values furnished by Delaunay's great work; and to examine whether the equations are thereby satisfied. With painful alarm, I find that they are not satisfied; and that the discordance, or failure of satisfying the equations, is large. The critical trial depends on the great mass of computations in Section II. These have been made in duplicate, with all the care for accuracy that anxiety could supply. Still I cannot but fear that the error which is the source of discordance must be on my part. I cannot conjecture whether I may be able to examine sufficiently into this matter.

The work consists really of two parts, that which is based on spherical form of the earth, and that which is given by the oblateness of its real form The alarm which I have expressed applies solely to the first of these On the second, though its results differ from those of some astronomers, I have no fear of error Of other chapters it is not necessary to speak

G B AIRY

The White House, Greenwich,
1886, August 18

The proposed LUNAR THEORY was first brought before the Public by the following address to the ROYAL ASTRONOMICAL SOCIETY at their Meeting of 1874, January 9

In placing before the public a somewhat novel form of treatment of the Lunary Theory, it appears desirable to introduce the explanation of the method now proposed by a rapid survey of the methods hitherto employed

In the whole range of physical mathematics, there is perhaps nothing more remarkable than the beauty of the geometrical integrations, in the III Book of *Newton's Principia*, for the Lunar Inequalities in Latitude and for the Lunar Variation, and the general accuracy of the results It is clear also, from a few remarks in the 11th Section of the I Book, and from an unexplained remark on the comparison of the inequalities of the satellites of other planets with those of our Moon in the III Book, that Newton perfectly understood the origin of what are now called the terms of the second order, by which the Velocity of Progression of the Apse, and the Evection, are so much increased But Newton published no numerical calculation of those quantities, and the theory was, so far, left imperfect A more powerful calculus was necessary

The want was supplied by the Differential Calculus, in the shape in which it was established among Continental Mathematicians, and the particular form in which it was applied by Clairaut to the Lunar Theory exhibited at once the power of the Calculus and the ease of applying it The simple form of Clairaut's differential equations for parallax and latitude opened out the entire process of extending the theory to any degree of accuracy, and showed at the same time the steps by which periodical inequalities of one form are deduced from the combination of periodical inequalities of other forms I think that scarcely sufficient honour has been given to Clairaut for the formation of this special equation, without which the progress of the theory would probably have been very slow Even now it is the best form in which a beginner can enter upon the studies of the Lunar Theory Clairaut's theory gives the time in terms of the arc of longitude described, which is not without advantage in the treatment of equations of long period, but it requires a final reversion of series, in order to give the longitude in terms of the time Mathematicians in the later part of the present century have preferred a form in which the Moon's ordinates are expressed immediately in terms of the time I give my adhesion to this method, but at the same time I am anxious to offer my testimony to the value of the process so successfully introduced at a most critical point in the progress of the science

The next important extensions of the theory were those of Laplace and Damoiseau both founded on Clairaut's equation, both exhibiting the subordinate equations derived from the comparison of co efficients which are expressed by unexpanded algebraical fractions whose denominators are very complicated (the piles of these fractions, especially in Damoiseau's work, are appalling), both giving the first results in numerical values for the co-efficients of numerous arguments which are multiples of longitude, both leaving in great obscurity the process by which the numerical solutions of these algebraical comparisons were obtained, and both giving the final results in terms depending on the time Damoiseau, however, added to this investigation a work which demands our gratitude a system of Lunar Tables expressly founded on the aggregation of simple periodical terms having for arguments different multiples of the time It was by use of these Tables (with small additions derived principally from Plana) that I conducted the

great Reduction of Lunar Observations from 1750 to 1853, and deduced from them the corrections of the principal co efficients Damoiseau's angular values were all expressed in the centesimal division of the quadrant a method which possesses so many advantages that I hope for its adoption in future tables

Plana's work, which followed, was not entirely pure in its method It commences, for instance, with an application of theorems for the "variation of constants," here introduced with great advantage But in the more advanced parts it may be described as established on the use of the time as the independent variable, and as exhibiting every co-efficient in a series of algebraical terms without denominators Viewed as leading to an algebraical result, this work was a great advance beyond all which had preceded it, and in numerical accuracy it is probable that something was gained

I do not advert to the extensive investigations of Lubbock, because they were principally in the nature of verifications, adopting generally M Plana's system Nor do I consider the important questions raised by Professor Adams, because they are, in fact, a re-examination of specific points in a received theory Professor Hansen's theory and tables require mention, principally in explanation of my reasons for almost omitting them from a view of the progress of the science I attach the highest value to Professor Hansen's discovery of two inequalities in longitude produced by *Venus*, of which one is universally accepted, and the other, though controverted, still appears plausible And I value the new equation which he introduced in the Moon's latitude I believe also that the object which Professor Hansen originally proposed to himself, namely, the more rapid convergance of terms, has been (in some measure at least) attained Yet I think that the general form of his theory, differing so much from the two systems which had preceded it, and presenting little facility for correcting elements from observations, is so far objectionable that it is not likely to be adopted by future lunar theorists, and that its introduction was, in fact, a retrogade step But, in common with all who are practically concerned with lunar observations, I am grateful for his Lunar Tables, which, embodying the results of his own theory and the Greenwich corrections of elements, and published at a time when the existing tables were running wild, have been most beneficial to practical science

But there remains one glorious work, almost superhuman in its labour, and perfect beyond others in the detailed exhibition of its results, the Lunar Theory of Delaunay In this the time is adopted as the independent variable The masses of undeveloped fractions here exhibited are greater than those of Damoiseau, the development in terms without denominators is more extensive than that of Plana, and the numerical evaluation of every term is more complete than that of any preceding writer Some terms to which we should have attached great interest are lost (at least for the present) by the untimely death of M Delaunay

Now in all these works, so far as I have remarked, the following characteristics hold —

(1) Each investigator has begun his work *de novo*, without making any use of the results of preceding investigators, even with the application of contingent corrections

(2) Each investigator has used the fractions, in symbolical terms, to which I have alluded, and, by adherence to the symbolical form, has been compelled to expand them in series with rapidly increasing co-efficients

(3) The nature of the steps has compelled the investigators to decide the succession of their terms, not by numerical magnitude, but by algebraical order. And this has produced great inequality of convergence. Delaunay's smaller co-efficients are probably correct, as he has exhibited them, to 0″.0001, but his larger terms converge so slowly that he has been compelled to supplement them by an assumed law of decrease, and they may perhaps be in error by almost 1″.0000.

(4) The mental labour in these operations is fearfully great. M. Plana once remarked to me, "Quelquefois, Monsieur, ces calculs me font presque perdre la tête."

(5) This labour cannot be alleviated, even in the examination of work done, by an amanuensis or assistant.

In consideration of these circumstances (which I have known, as well from examination of the works of others, as from my private investigations), I have long held the opinion that a Lunar Theory, in which every co-efficient is expressed, from the very beginning of the process and throughout, by simple numbers, is very desirable. My ideas on this subject have by degrees assumed an orderly form, and I am now able to exhibit their leading points, as follow:—

(1*) I propose to assume Delaunay's final numerical expressions, for longitude, latitude, and parallax, with the addition of secular equations, as my fundamental numbers. These will be converted into other numerical expressions referred to more convenient units. To every number, as far as I think necessary, will be attached a symbolic term for contingent correction, in some cases considered as varying with the time. In all cases I assume that this correction will be so small that its first power will be sufficient. The secular terms will probably introduce cosines with sines of the same argument.

(2*) I propose to substitute these numbers with symbolical corrections in the equations in which the time is adopted as independent variable. The fractions to which I have alluded will still occur, but not in a troublesome symbolical form. The greatest complication of denominators will be that of "a number with small symbolical correction attached to it," which will be instantly converted into two terms without denominator. There will never be an infinite series.

(3*) The order of terms will be numerical, and, as far as I perceive, they will be equally accurate throughout.

(4*) The details of the work will be very easy.

(5*) A great part of the work can be intrusted to a mere computer, and probably the whole can be examined, or can be repeated in duplicate, by such assistant.

To this I add—

(6*) I have strong confidence that equations of very long period may thus be examined with great severity, especially when there is reason to suspect that the form of the principal arguments may be slightly changed.

(7*) The result of the comparison of the terms in the mechanical or gravitational equations will be, a great number of equations for determining the numerical values of a great

numbei of small quantıtıes I antıcıpate no dıfficulty ın the solutıon, ıt ıs usually sufficıent for the determınatıon of any one of the small quantıtıes, to change (where necessary) the sıgn of ıts co-efficıent, so as to have all ıts co-efficıents wıth the same sıgn (the sıgn of the constant term beıng also changed), and to add all, neglectıng all the other unknown quantıtıes In some cases, however, ıt may be necessaıy to tıeat two of these correctıons ın combınatıon

Though very late, I have actually begun a Lunar Theoıy ın the shape whıch I have descrıbed

NUMERICAL LUNAR THEORY

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NUMERICAL LUNAR THEORY.

INTRODUCTION

The principle of this Theory is, that the formulæ or numbers defining for any moment the position of the Moon, which have been obtained by Delaunay from algebraical theory, are used as a very approximate basis, but that every term is supposed to admit of a correction, so small that its square will never be sensible, and that, in prosecuting the algebraic treatment of the Equations of Lunar Theory, this correction is to be expressed as an Algebraical Symbol peculiar to each term, called the "Symbolical Variation" of that term, but that its real value applicable to each term is ultimately to be obtained in a Numerical form For this purpose, the work must be carried through the following successive steps —

1 The algebraical form of the Lunar Theory must be given This, as in all theories of the motion of a free body in space, will consist of three independent Fundamental Equations [which, at the end of Section I and through the work, will be distinguished as Equation (10), Equation (11), Equation (12)] Each of these equations will consist of two parts

The first side (which we shall call the *orbital* side) contains algebraical and numerical quantities, which consist only of combinations and differentials of the assumed (Delaunay's) formulæ and co efficients that define the tabular place and motion of the Moon, accompanied with multiples of the Symbolical Variations of the possible additions to these numerical quantities

This *orbital* side, formed by combinations of the differentials which originate in the simplest steps of algebraical mechanics, represents the force which must be provided in order to maintain the assumed motion in the assumed orbit (as symbolically corrected)

The second side (or *gravitational* side) exhibits the forces which are provided to maintain the orbital motion, it contains the results of the mutual attraction of the Sun, Earth, and Moon, with other multiples of the same Symbolical Variations, for the change of forces which will be produced by the change in the value of co-ordinates arising from the possible addition to the orbital quantities, and also with any required additions (when intruding forces are considered) that are not included in the usual expressions of terrestrial and solar forces

The assertion (by the symbol of equality =) that the orbital side is equal to the gravitational side, or the assertion "Orbital Side − Gravitational Side = o," completes the Fundamental Equation

The factors of the Symbolical Variations will be treated in detail in Sections VII. and VIII

2 The first expressions of the co ordinates, both of the Moon and of the other attracting bodies, and also the expressions for the forces of their mutual action, are necessarily formed

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in terms of the true co-ordinates of the several bodies, in the first instance, in terms of the rectangular co-ordinates x, y, z, and subsequently, in terms of true radius vector, true longitude referred to a given plane, and true latitude referred to the same plane. In this state, the Equations (10), (11), (12), are exhibited at the end of Section I. Some simple constant numbers there entering (applying to the Sun's mean distance) are taken by anticipation from a following section. For the solution of these equations, it is necessary to express all by reference to one fundamental variable. The variable adopted by Laplace was "the Moon's true longitude". But Plana and others, down to Delaunay, have adopted "the time", and this is followed in the present work. It is necessary, for this purpose, to employ Delaunay's expressions in terms of the time (in very long series of different periodic terms with different arguments and different co-efficients) for the radius vector, for the longitude, and for the latitude, of Sun, Earth and Moon, and to perform upon them the various operations of multiplication, differentiation &c. which are indicated in Equations (10), (11), (12). The mode of conducting this work is explained in Section II, Part 1.

The results for the orbital sides of the Fundamental Equations are exhibited in Section II, Part 2 (for Equations (10) and (11)), and in Section II, Part 3 (for Equation (12)).

The similar conversion of the gravitational side is effected in several successive Sections and their Parts. Part 1 of Section III contains the operations for the terrestro-lunar force in Equation (10), and Part 2 contains those for Equation (12), both requiring a factor M depending on the proportion of the masses of the Earth and Moon. (It will be seen in the course of the investigation that the terrestro-lunar forces contribute nothing to Equation (11).) Part 1 of Section IV consists of algebraical investigation of the Solar Forces, Part 2 exhibits the converted forces for Equations (10) and (11), and Part 3 those for Equation (12). The calculations for the Solar Forces in the plane of the ecliptic are very long, and the steps of calculation are therefore exhibited only for the first fifteen arguments.

The process which has been used for the verification of these numerical operations, and of which one result is a small change of the assumed value of parallax, is described in Section V.

All parts of the Three Fundamental Equations are now expressed by periodical terms, whose arguments are given in multiples of time, and whose co-efficients are numerical. The co-efficient M only retains the symbolical form.

3. Antecedent investigations (not cited here in detail) have shown that terms of expression similar to Delaunay's, with perpetually-diminishing co-efficients, are competent to produce a lunar theory founded on the principle of gravitation, to any assigned degree of accuracy. Therefore, if Delaunay's numerical calculations are correct, the substitution of every one of his terms in each of the three "Equations," without alteration of his co-efficients, will give for each of the "Equations" the result 0. The examination of the possibility of satisfying this requirement must be conducted by the following steps.

First, an exact or approximate value of M must be determined, which will produce, in every one of the subordinate terms that contain M, the results, "Terms of Equation (10) = 0," "Terms of Equation (12) = 0." If no satisfactory value of M is found, the value that is judged most probable must be adopted for use, with the symbol δM attached, in order to give

means of making a small variation, if necessary This investigation, and the substitution of an assumed value for M, and the exhibition of the outstanding errors of the three "Equations," occupy Section VI

4 The important object to which investigations are now to be directed is, the expression of the outstanding discordances for each of the three "Equations," in multiples of variations of the co-efficients of every term in those "long series" to which allusion has been made in Article 2 For this expression we must pass through two steps, of which the first only is treated in this Article The first exhibition of Symbolical Variation of the "Equations" must consist of multiples of the simple symbols for Variations, of Ecliptic Parallax (or reciprocal Radius-Vector), of Longitude, and of Normal to the Ecliptic-Plane The expressions for the multipliers of these symbols are to be formed by a differentiating-process performed upon the algebraic expressions for the "Equations" The details of this operation are given fully in Section VII From these, the First Factorial Table is formed, exhibiting (by means of those multipliers, and by collection of results applying to each Orbital Element) the serial factors which are to multiply the Variations of the Orbital Elements

5 The second or final step for exhibiting Symbolical Variation of the "Equations" is the following —The expressions for longitude, ecliptic parallax, and ecliptic normal, are now to be introduced in the form of those long series of terms mentioned in Article 2, where each term is a numerical co-efficient multiplying an algebraical periodical quantity, the lengths of period for all the different quantities being different The Variations of longitude, &c are to be expressed by Variations of those co efficients and of the arguments of the periodical quantities, and are then to be multiplied by the Serial Factors mentioned at the end of Article 4 This completes the detailed exhibition of Symbolical Variations of the Fundamental Equations

6 Every combination of these Symbolical Variations, multiplying a term distinguished by any one argument, is to be used entirely separated from terms connected with any other argument, and is to be multiplied by a separate indeterminate co-efficient, to correct the numerical outstanding value of a separate portion (based on the same argument) of each Fundamental Equation Thus every variation of subordinate co-efficient or argument furnishes a separate portion of one of the Fundamental Equations, and, by means of these, the value of every indeterminate co efficient can be found, and every tabular element can be corrected The steps leading to this will be found in Section IX As the process is exceedingly complicated, it may be necessary to divide it into parts, for which no general rule can be previously laid down Thus, if the discordances seem to indicate the existence of great errors in individual co efficients, it may be best to commence with them, otherwise it may be best to begin with δM, and with the motion of apse, and with the motion of node (The two latter methods are not employed here)

This terminates the ordinary Centripetal Lunar Theory

The following investigations are important, but they will be best treated as appendages to the Centripetal Lunar Theory

7 Considerations on the determination and application of the numerical value of the Moon's Mass are suggested for examination

8 The terms produced by the Oblateness of the Earth are to be considered as in the class of Intruding Forces in the "Equations," and their results may be obtained by use of the Modified Factorial Table

9 The terms which depend on the slow change in the position of the Solar Ecliptic may be treated in the same manner

10 The terms which depend on the slow change of Excentricity of the Solar Orbit may be similarly treated

11 Terms depending on other external causes, not treated in this work, may in all cases be referred to a discussion similar to that which terminates in Article 6 (above) They present no difficulty in the Lunar Theory, strictly so called, their real difficulties consist in the preparation of the first formulæ expressing the first mechanical effect of those external causes

NUMERICAL LUNAR THEORY.

SECTION I.

ALGEBRAICAL FORM OF THE THEORY.

FUNDAMENTAL EQUATIONS (10), (11), (12)

SECTION I—ALGEBRAICAL FORM OF THE THEORY

It is assumed that the motion force of one body acting upon another, and the movement of the body subjected to such motion-force, may be resolved into the directions of three rectangular co ordinates, by using as factors the cosines of the angles made, by the direction of the motion force or of the movement, with those three co ordinates, and that independent equations may be formed for those three directions by equating the motion-force in each direction to the second differential co efficient (with regard to time) of the ordinate in that direction

It is also assumed that, in the instance of gravitation, the motion-force, which the attraction of one body produces on each of the material particles of another body, is entirely independent of the existence of other material particles of that body, it being understood, nevertheless, that there may be connexions, or attractive or repulsive forces, between those material particles, whose actions and reactions on each particle are to be combined with the above mentioned forces produced by extraneous action

Finally, it is assumed, as a law special to gravitation, that each body attracts each other body in the direction of the line joining the two bodies, and with a motion-force inversely proportional to the square of the distance between the bodies, and directly proportional to the mass of the attracting body

Let σ, ϵ, μ, be the masses of the Sun, Earth, and Moon, estimated by the acceleration which they respectively produce on an extraneous particle at distance 1, in time 1 The unit of distance will be left arbitrary, the unit of time will be mentioned in Section II

All ordinates are to be understood as referred to the plane of position of the ecliptic at some definite time (as the year 1900), and a line drawn in the direction of the first point of Aries in that year is the origin of longitudes or angles in that ecliptic plane For rectangular co ordinates x is directed to the first point of Aries in that year, y at right angles to x, in the plane of the ecliptic of that year, on the side corresponding to "direct" motion of the Moon from Aries, and z normal to that plane, towards the north

For the position of the Sun in the orbit which he appears to describe round the centre of gravity of the terrestro-lunar system let A be his mean distance from that center of gravity, E the excentricity of his orbit, supposed invariable (the effects of a small error in this supposition will be corrected as suggested in Introduction, Article 10), R his distance at any moment, V his apparent longitude at the same time, as viewed from that center of gravity $W = V + 180°$, the apparent longitude of the center of gravity as viewed from the Sun For the present, it is assumed that the Sun appears to move exactly in the plane of the ecliptic of the year 1900, and therefore has no apparent latitude (the treatment of the effects of a small error in this assumption is noticed in Introduction, Article 9)

For the position of the Moon with respect to the Earth let a be the mean distance of the Moon from the Earth, r the distance at any moment, l the northerly inclination of that distance to the ecliptic plane, or the Moon's latitude, $r \cos l$ the projection of r upon the plane

of the ecliptic for which we shall sometimes write ρ, v the angle made by ρ with the line to the first point of Aries, or the Moon's longitude It will be seen that $x = \rho \cos v$ or $r \cos l \cos v$, $y = \rho \sin v$ or $r \cos l \sin v$, $z = \rho \tan l$ or $r \sin l$

For the position of the Moon with respect to the center of gravity of the terrestro-lunar system, the same formulæ will apply, requiring only the substitution of $r \times \frac{\epsilon}{\epsilon + \mu}$ for r and $\rho \times \frac{\epsilon}{\epsilon + \mu}$ for ρ For the position of the Earth with respect to the center of gravity, it is necessary to substitute $r \times \frac{\mu}{\epsilon + \mu}$ and $\rho \times \frac{\mu}{\epsilon + \mu}$ for r and ρ respectively, and also to change the signs of the three resulting terms

The forces which affect the relative motions of the Earth and Moon are derived from three sources (1) The mutual attraction of the Earth and Moon, which we shall call "Terrestrio-Lunar Attraction" (2) The excess of the Sun's attraction on the Moon above its attraction on the Earth, which we shall call "Solar Attraction" (3) Small modifications of those forces (of which two have been mentioned above, and a third, the effect of Earth's oblateness, is noticed in Introduction, Article 8), or small extraneous forces, in either case so small that it will be unnecessary to consider the squares of their numerical representatives we shall call these "Small Additional Forces" The first and second of these classes of force will be most conveniently represented by forces referring to the relative motion of the Moon round the Earth as Center, and estimated numerically,—in the direction of ρ,—in the direction transversal to ρ in the ecliptic plane,—and in z towards the north The forces of the third class must be reduced to the same form, and may be called respectively, P, T, and Z we shall have no occasion to refer to them until we enter on the investigations of the subjects mentioned in the Introduction, Articles 8, 9, 10

In our algebraic investigations of all these forces, we shall commence with ecliptic forces in the direction of ρ, ecliptic forces transversal to ρ, and forces normal to the ecliptic, and shall convert them into rectangular forces in the directions x, y, z Forming the mechanical equations with respect to x, y, and z, we shall deduce from them the rectangular equations which apply to the longitudinal measure of the ecliptic radius ρ,—to the double area in the plane of the ecliptic,—and to the measure of z normal to the ecliptic plane We shall then reconvert these rectangular equations into equations depending on —force in direction of ρ,—force in ecliptic plane transversal to ρ,—force normal to the ecliptic Call these three forces (ρf), (tf), (zf) Converting (ρf) and (tf) into rectangular forces (xf), (yf) —

$$+ (xf) = + (\rho f) \, \tfrac{x}{\rho} - (tf) \, \tfrac{y}{\rho}, \qquad + (yf) = + (\rho f) \, \tfrac{y}{\rho} + (tf) \, \tfrac{x}{\rho},$$

and the equations of motion in x and y are,—

$$(1) \; + \frac{dx}{dt} = + (\rho f) \, \frac{x}{\rho} - (tf) \, \frac{y}{\rho}, \qquad (2) + \frac{dy}{dt} = + (\rho f) \, \frac{y}{\rho} + (tf) \, \frac{x}{\rho}$$

The equation of motion in z is simply—

$$(3) \; + \frac{dz}{dt} = (zf)$$

By a first combination of (1) and (2),

$$x\frac{d^2x}{dt^2} + y\frac{d^2y}{dt^2} = +(\rho f)\,\frac{x^2+y^2}{\rho},$$

$$\text{Or } + \left\{x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + y\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^2\right\} - \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} = +(\rho f)\,\rho,$$

$$\text{Or } + \frac{d}{dt}\left\{x\frac{dx}{dt} + y\frac{dy}{dt}\right\} - \left\{\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{dv}{dt}\right)^2\right\} = +(\rho f)\,\rho$$

But $x\frac{dx}{dt} + y\frac{dy}{dt} = \rho\frac{d\rho}{dt} = \frac{1}{2}\frac{d}{dt}(\rho^2)$, and $\frac{d}{dt}\left\{x\frac{dx}{dt} + y\frac{dy}{dt}\right\} = \frac{1}{2}\frac{d^2}{dt^2}(\rho^2)$

And the equation is now—

$$(4)\quad + \frac{1}{2}\frac{d^2}{dt^2}\left\{(r\cos l)^2\right\} - \left\{\frac{d}{dt}(r\cos l)\right\}^2 - (r\cos l)^2 \times \left(\frac{dv}{dt}\right)^2 = +(\rho f)\; r\cos l$$

This is the equation of Ecliptic Radius Vector

By a second combination of (1) and (2),

$$+x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = \frac{+x^2+y^2}{\rho}\,(tf),\ \text{or } \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = +(tf)\,\rho,$$

Or,

$$(5)\quad + \frac{d}{dt}\left\{(r\cos l)^2\,\frac{dv}{dt}\right\} = +(tf)\; r\cos l$$

This is the equation of Ecliptic Areas

In equation (3) we shall merely substitute a value for z

$$(6)\quad + \frac{d^2}{dt^2}(r\sin l) = +(zf)$$

This is the equation of Elevation above the Ecliptic

We have now to give values to the different symbols, (tf), (ρf), (zf), for the corresponding parts of the different origins of force

First, for Terrestro Lunar Attraction

The motion-force which the Earth exerts to draw the Moon towards the Earth is $\frac{\epsilon}{r^2}$ in the direction of the line that joins them, the motion force which the Moon exerts to draw the Earth towards the Moon is $\frac{\mu}{r^2}$ in the opposite direction of the same line, and the combined "Terrestrial and Lunar Attraction," which tends to draw the Moon relatively towards the Earth is $\frac{\epsilon+\mu}{r^2}$, which is to have the negative sign, inasmuch as it tends to diminish r The resolved part of this force transversal to ρ in the ecliptic plane is evidently 0, that in the direction ρ is $-\frac{\epsilon+\mu}{r^2}\cos l$, and that in the direction z is $-\frac{\epsilon+\mu}{r^2}\sin l$ It is now desirable to express $\epsilon+\mu$ (at least in a rude approximation) in terms of the elements specified at the beginning of this Section, with the addition of the periodic times of revolution

For the unit of time, it will be found extremely convenient to adopt "the Moon's mean periodic time divided by 2π," or $\frac{\text{Sidereal month}}{2\pi}$ If the Moon moved (relatively) in a circle whose radius is $\mathbf{a}$, the circumferential velocity (for unit of time) would be $\mathbf{a}$, the centripetal force at the circumferential distance (by the formula $\frac{(\text{velocity})^2}{\text{radius}}$) must be $\frac{\mathbf{a}^2}{\mathbf{a}} = \mathbf{a}$, and the centripetal force at distance 1 must be $\mathbf{a}^3$ And the same value would be found if the Moon moved, undisturbed, in an elliptic orbit in which the mean distance $= \mathbf{a}$

But the Moon's relative motion is disturbed by the action of the Sun, attracting both Earth and Moon, but in different degrees and different directions, according to their position relative to the Sun. The Sun's force being $= \frac{\sigma}{(\text{distance of attracted body})}$ and in the direction of the joining line, it is seen (by reference to Section IV., or by general reasoning), after due estimation of the separate actions on the two bodies and taking their difference, that the mean disturbing force transversal to the radius vector is 0, and that the mean disturbing force in the direction of radius is sensibly $= + 2\ \frac{\sigma a}{A^3}$ at syzygies, and $= - \frac{\sigma a}{A^3}$ at quadratures, and its mean value may be taken as $+ \frac{1}{2}\ \frac{\sigma a}{A^3}$, by which the mutual attraction of Earth and Moon is diminished. Hence the centripetal force at the circumferential distance, which (as we have found) = a, and which must consist of $\frac{\epsilon + \mu}{a^2} - \frac{1}{2}\ \frac{\sigma a}{A^3}$, gives the equation $a = \frac{\epsilon + \mu}{a} - \frac{1}{2} \frac{\sigma a}{A^3}$, or $\frac{\epsilon + \mu}{a^3} = 1 + \frac{1}{2} \frac{\sigma}{A^3}$.

Now considering the terrestrio-lunar system as revolving round the Sun in the time "year,", the measure of which, referred to the unit above mentioned is, Year $\times \frac{2\pi}{\text{Sidereal month}}$ (the numerical value is 83·9982, or 84 nearly), the circumference is $2\pi \times A$, the circumferential velocity is $A \times \frac{\text{Sidereal month}}{\text{Year}}$, and the centripetal force at circumference must be $A \times \left(\frac{\text{Sidereal month}}{\text{Year}}\right)^2$. This must $= \frac{\sigma + \epsilon + \mu}{A}$, or $\frac{\sigma}{A^2} + \frac{\epsilon + \mu}{A^2}$. Hence we find $\frac{\sigma}{A^3} = \left(\frac{\text{Sidereal month}}{\text{Year}}\right)^2 - \frac{\epsilon + \mu}{A^3}$. Substituting this in the former equation, $\frac{\epsilon + \mu}{a^3} = 1 + \frac{1}{2}\left(\frac{\text{Sidereal month}}{\text{Year}}\right)^2 - \frac{1}{2}\frac{\epsilon + \mu}{A^3}$, or $(\epsilon + \mu) \times \left(\frac{1}{a^3} + \frac{1}{2}\frac{1}{A^3}\right)$ $= 1 + \frac{1}{2}\left(\frac{\text{Sidereal month}}{\text{Year}}\right)^2$, and $\frac{\epsilon + \mu}{a^3} = \frac{1 + \frac{1}{2}\left(\frac{\text{Sidereal month}}{\text{Year}}\right)^2}{1 + \frac{1}{2}\left(\frac{a}{A}\right)^3}$. With the values, $\frac{\text{Sidereal month}}{\text{Year}}$ = ·0748013, and $\frac{a}{A} = \frac{\text{Sun's mean h. e. parallax}}{\text{Moon's h. e. parallax}} = \frac{8''\cdot 91}{3422''\cdot 30}$ = ·0026006, this becomes $\frac{\epsilon + \mu}{a^3}$ = 1·0027976. In this computation the effect of introduction of Solar Parallax is numerically insensible.

It is to be remarked that the process by which we have found the numerical value of $\frac{\epsilon + \mu}{a^3}$ is not severely accurate, and we must consider that number as liable to correction. In adopting (for that value of **a** which will lead to the best representation of the fundamental masses of the Earth and Moon), the mean value of r, we have been guided by the general conception that a mean value of r in every part of the orbit will produce the same general effect as to attractions, periodic times, effect of solar action, &c., as the combination of all values of r. And when the irregularity of values is very small, there appears to be no doubt that this will be sensibly true. But there is not the same security when the excentricity and other causes of perturbation are as great as they are in the terrestro-lunar system. It appears possible, for instance, that we may more advantageously adopt the symbol α (to be used frequently hereafter), which represents a radius that produces a value of parallax of the Moon equal or nearly equal to the mean value of parallax. (It will be seen in Section II. that $\mathbf{a} = \alpha \times$ 1·0015647.) The substitutions in our equations will decide on this point: meantime we remark that it is very desirable to obtain a fairly correct approximate value as early as possible, in order that the ultimate correction to it may be so small as not to endanger our fundamental principle, "that squares and products of corrections shall be too small to be recognized in further operations." Putting M for $\frac{\epsilon + \mu}{a^3}$, we may be assured that the numerical value of M will not differ materially from 1.

And thus, for completing the several Equations on page 10 —

First, for Terrestro Lunar Attraction,

The force in (ρf) is $-\frac{\epsilon+\mu}{r^2}\cos l$, and the term contributed to Equation (4) is $-\frac{\epsilon+\mu}{r}\cos^2 l$

The force in (tf) is 0, and nothing is contributed to Equation (5)

The force in (zf) is $-\frac{\epsilon+\mu}{r}\sin l$, and the term contributed to Equation (6) is $-\frac{\epsilon+\mu}{r^2}\sin l$

Second, for Solar Attraction

For the details of this, we must refer to the end of Section IV, Parts 2 and 3, pages 66 and 67 It will be remarked, in pages 54 and 55, that the numbers on pages 66 and 67 exhibit the difference of the solar effects on the Earth and on the Moon

Third, for Small Additional Forces

The terms contributed to the Equations will be,

To Equation (4), $+P \cdot r \cos l$

To Equation (5), $+T \cdot r \cos l$

To Equation (6), $+Z$

Combining these with the principal terms at the head of page 10, our Equations will take the following form —

$$\text{Equation (10)} \left\{ \begin{array}{l} +\left\{\frac{d}{dt}\left(\frac{r}{a}\cos l\right)\right\}^2 + \left\{\left(\frac{r}{a}\cos l\right)^2 \left(\frac{dv}{dt}\right)^2\right\} - \frac{1}{2}\frac{d^2}{dt^2}\left\{\left(\frac{r}{a}\cdot\cos.\, l\right)^2\right\} \\ \qquad -\frac{\epsilon+\mu}{a}\cdot\frac{a}{r}(\cos l)^2 \\ + \text{terms produced by Solar Attraction, Section IV, Column 64} \\ +\frac{P}{a}\cdot\frac{r}{a}\cos l \end{array} \right\} = 0$$

$$\text{Equation (11)} \left\{ \begin{array}{l} -\frac{d}{dt}\left\{\left(\frac{r}{a}\cos l\right)^2 \frac{dv}{dt}\right\} \\ + \text{terms produced by Solar Attraction, Section IV, Column 68} \\ +\frac{T}{a}\cdot\frac{r}{a}\cos l \end{array} \right\} = 0$$

$$\text{Equation (12)} \left\{ \begin{array}{l} -\frac{d^2}{dt^2}\left\{\frac{r}{a}\sin l\right\} - \frac{\epsilon+\mu}{a^2}\left(\frac{a}{r}\right)^2\sin l \\ + \text{terms produced by Solar Attraction, Section IV, Column 72} \\ +\frac{Z}{a} \end{array} \right\} = 0.$$

For effecting a solution of these Equations to a high degree of accuracy, it is necessary to substitute, for the several symbols, assumed numerical values, advanced to the best approximation which the investigations of preceding theorists, as well as our own examinations, justify us in adopting, each value being understood as accompanied with a symbol of contingent correction And the equations thus formed are to be so arranged that the numerical corrections of the symbols can be determined and applied to the assumed numerical values

The assumed values adopted here are those given by Delaunay, Expression Numérique, attached to the *Connaissance des Temps*, 1860, *Appendix*,* the numbers being all altered here in the proportion of 3422·7000 to 1·0000000 (the leading numbers for parallax in the two systems of numerical expression). But the adoption of these numbers, for the present work, is mere matter of numerical convenience, not alluding in any degree to the specialities of Delaunay's Theory.

We shall now modify our algebraical forms of the geometrical relations, so as to produce more symmetrical expressions in reference to the Earth and the Moon. And we shall omit, for the future, the terms P, T, and Z, depending on accidental forces, unless they should be required for special perturbations.

For the Origin of Co-ordinates, we now adopt the Center of Gravity of the Earth and Moon. The symbols r and v have the same meaning as before, but originate at a point very slightly different from the former origin. All details of the results of this change will be found in Section IV., Part I., page 52, and following investigations.

* *Erratum* in Delaunay's *Appendix*, page 31, third line from the bottom. For $-0''\cdot1960 \cos 4D$, read $+0''\cdot1960 \cos 4D$. See Delaunay's *Théorie*, Tome II., pp. 582 and 921.

NUMERICAL LUNAR THEORY.

SECTION II. PART 1

FORMATION OF ORBITAL QUANTITIES.

NUMERICAL LUNAR THEORY

Section II.—Part 1.—Formation of Orbital Quantities

In the Introduction it has been explained that very approximate expressions, with numerical co-efficients and numerical factors of arguments, for three elements defining the moon's orbital place, are to be adopted, as subject to very small corrections (which it is the object of this work to investigate), and that, with these very approximate expressions, we are to form two sets of quantities, one set representing geometrical functions of the moon's orbital place and motion, and the forces *required* to maintain them, the other set giving values of the forces which pure mechanical considerations of the gravitational actions of other bodies on the moon indicate as *really acting* on her, the ultimate solutions resting on the comparison of the two sets of quantities. These geometrical and mechanical parts are, in fact, the first and second parts respectively of the three Equations (10), (11), (12), and in this Section we proceed to form the first parts of those three equations, by use of three adopted expressions for the defining elements.

In selecting the very approximate expressions of the three elements, there could be no hesitation in fixing on Delaunay's expansions for Parallax, Longitude, and Latitude, both because their accuracy is generally accepted, and because the form in which they are given is admirably adapted to extension of the theory or to correction of the co-efficients.

We may remark here, as has been said in the Introduction, that there is no doubt of the competency of our adopted form to represent the motions founded on the primary conceptions of gravitational mechanics, provided that proper numerical values are given to the co-efficients and factors of arguments. In the ordinary methods of proceeding, the substitution of large terms produces small terms with different arguments, which lead to closer solutions of the equations, these produce still smaller, leading more nearly to exact solution, and so proceeding to any assigned degree of accuracy, and all these terms are terms of the form employed in our investigations here. But the process is liable to accidental and numerical error, which it is our object now to correct. For any other small mechanical forces, the special effects can be computed without difficulty.

We shall premise a few words on the values to which all numerical expressions are referred.

Such expressions as $\left(\frac{a}{r}\right)^n$, where both a and r are the values of linear measures, are evidently to be referred to the abstract 1, and it is also convenient to refer the angles, considered in the first instance as arcs in a circle, to the radius of that circle, and therefore to refer "angle" or $\frac{\text{arc}}{\text{reading}}$ to the abstract 1, as unit of angle. In fixing on the extent of decimal sub-division to which the numerical expressions should be carried, it was decided to adopt the limit 0.0000001 or 10^{-7}. This secondary unit corresponds, in angle, to one fiftieth of a second nearly. The value of this secondary unit being always borne in mind, there is no necessity for writing the cyphers preceding the significant figures, as in ordinary decimal computation. In some instances, it appears necessary to proceed to 10^{-8} or 10^{-9}, in these cases the additional decimals are separated by a comma.

In the operations which commence at this point, we shall have to effect a great number of multiplications or divisions of decimal expressions by decimal expressions, the factors and the product or quotient being almost universally included between the place of units and the place of seventh decimals For a convenient and secure process, which would not employ unnecessary figures, and would leave no uncertainty on the decimal point, the method of successive divisions, for forming successive additions or subtractions, appeared best There the multiplicand is divided by a convenient divisor (usually of one figure, sometimes of two) to form the first quotient, approximating to the quotient sought , that first quotient is divided by a convenient divisor, for formation of a second quotient (additive to or subtractive from the first quotient) to produce a nearer approximation to the product sought the second quotient is similarly divided to form a third, and so on Sometimes (instead of a divisor), a multiplier, divided by 10, 100, &c, is convenient Thus, a factor, by which many terms are to be multiplied, is 545094 A process is tentatively found, by which 10000000 will be converted into 545094

			1 0000000
+	$\frac{1}{20}$	=	500000
+	$\frac{1}{10}$	=	+50000
			550000
−	$\frac{1}{10}$	=	−5000
			545000
+	$\frac{1}{60}$	=	+83,3
+	$\frac{1}{7}$	=	+11 9
			545095 2
−	$\frac{1}{10}$	=	−1,2
			545094

These multipliers, $+\frac{1}{20}$, $+\frac{1}{10}$, $-\frac{1}{10}$, $+\frac{1}{60}$, $+\frac{1}{7}$, $-\frac{1}{10}$, once ascertained, can be applied to any number which is to be multiplied by 545094 If, for instance, we wish to multiply 99812 by 545094 The process is the following —

			99812
+	$\frac{1}{20}$	=	4990 6
+	$\frac{1}{10}$	=	499,1
			5489 7
−	$\frac{1}{10}$	=	−49,9
			5439,8
+	$\frac{1}{60}$	=	+,8
+	$\frac{1}{7}$	=	+ 1
			5441

If we had to perform the opposite process, to divide by 545094 (instead of multiplying), the immediate operation would have been to convert 545094 into 10000000 (instead of converting 10000000 into 545094), and the method of obtaining a result would have been exactly similar

This method has been employed for the principal part of the multiplications which follow. In some instances the operation has been verified by ordinary multiplication, or by "contracted multiplication," or (occasionally) by logarithms.

We proceed now with the preparation of the new expressions for the Moon's co-ordinates, including (where necessary) the application of the processes for multiplication.

For Equatoreal Parallax, Delaunay's series is slightly modified to produce Sine of Parallax, and it becomes $3422''.7$ + terms with other arguments. It is convenient to divide the entire series by 3422.7, so that the co-efficient of the first term will be 1.0000000. This is done by the process described above, using the multipliers, $\frac{1}{3}$, $-\frac{1}{9}$, $-\frac{1}{10}$, $-\frac{1}{9}$, $-\frac{1}{30}$, $-\frac{1}{30}$.

This series of results for Assumed Parallax, deprived of its first term 1.0000000, is given in Column 1 of the following Tables (Section II, Part 2). In succeeding operations, these numbers of Column 1 are cited by the general symbol g with the "Reference for Argument" as subscript, thus $g_2 = +545095$, $g_3 = +99813$.

The square of $\left(\frac{a}{r} - 1\right)$ is then formed by multiplying every term of $\left(\frac{a}{r} - 1\right)$ (Column 1 just found) by every term of the same series. The process of multiplication described above, or any equivalent process, is to be used throughout for multiplication of the numerical co-efficients. For multiplication of the periodical terms, it is necessary to observe that every subordinate product here may be represented as $p \cos \overline{\Pi} \times q \cos \overline{\Phi}$, or $pq \cos \overline{\Pi} \cos \overline{\Phi}$. The product of $\cos \overline{\Pi} \cos \overline{\Phi}$ is evidently $\frac{1}{2} \cos \overline{\Pi - \Phi} + \frac{1}{2} \cos \overline{\Pi + \Phi}$, producing arguments which are different from those of the two factors. (In subsequent combinations, similar considerations apply to the products of sines by sines, or sines by cosines.) All the co-efficients of each new argument, by whatever multiplication produced, must be collected and added together, with former co-efficients of the same argument, if such exist. In some instances it is necessary to collect more than ten such co-efficients, produced by different multiplications.

The series for $\left(\frac{a}{r} - 1\right)^2$, being thus formed, is multiplied in the same manner by the series for $\left(\frac{a}{r} - 1\right)$ to form $\left(\frac{a}{r} - 1\right)^3$, by another similar multiplication $\left(\frac{a}{r} - 1\right)^4$ is formed, and so on. Then, by application of the binomial theorem, the values of $\left(\frac{a}{r}\right)^k$ or $\left\{1 - \left(\frac{a}{r} - 1\right)\right\}^k$ are formed for different values of k. Thus all numbers are prepared as far as Column 8.

For formation of $\sin \overline{\mathrm{l}}$ and powers of $\cos \overline{\mathrm{l}}$, the first step is to convert Delaunay's value of the Moon's latitude l, expressed in seconds, into terms of the new secondary unit. Since $100000'' = 0.4848137$, it was necessary to find successive factors for converting 100000 into 4848137. The adopted factors are $-\frac{1}{2}$, $-\frac{3}{100}$, $-\frac{1}{90}$, $+\frac{1}{20}$, $-\frac{1}{8}$. The complete transformed series for l is contained in Section II, Part 3, Column 24; its terms are cited by the general symbol k with the "Reference for Argument" as subscript, thus $k_{301} = +895027$, $k_{303} = +48978$, &c. The series for l being thus obtained, the successive series for the powers of l are obtained in the same manner as those for the powers of $\left(\frac{a}{r} - 1\right)$, and then the series for $\sin \overline{\mathrm{l}}$ and for the powers of $\cos \overline{\mathrm{l}}$ are found from the ordinary formulæ $\mathrm{l} - \frac{\mathrm{l}^3}{1 \cdot 2 \cdot 3} + \frac{\mathrm{l}^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ and $1 - \frac{\mathrm{l}^2}{1 \cdot 2} + \frac{\mathrm{l}^4}{1 \cdot 2 \cdot 3 \cdot 4}$ and the powers of the latter. The series for $\sin \overline{\mathrm{l}}$ is given in Column 27, and those for $\cos \overline{\mathrm{l}}$ and $\{\cos \overline{\mathrm{l}}\}^2$ in Columns 11 and 12.

For $\frac{r}{a}\cos\overline{|\mathrm{I}|}$, every term of the series for $\frac{r}{a}$ is multiplied by every term of the series for $\cos\overline{|\mathrm{I}|}$, and for $\left\{\frac{r}{a}\cos\overline{|\mathrm{I}|}\right\}$, every term of $\frac{r}{a}\cos\overline{|\mathrm{I}|}$ is multiplied by every term of the same series, exactly as in the operations for $\left(\frac{a}{r}-1\right)$

In Delaunay's expression for the longitude v, the co-efficient of every term is expressed in seconds of arc For our purposes, these co-efficients are converted into multiples of our secondary unit, in the same manner as the co-efficients in the expression for latitude, above These converted co-efficients are contained in Column 15, and are cited by the symbols h_2, h_3, h_4, &c

To form $\frac{dv}{dt}$, Section II, Part 2, Column 16, the following considerations are necessary If one term of $v = p \sin\overline{|\Pi|}$, p being a numeral, $\frac{dv}{dt}$ for that term will $= p \cos\overline{|\Pi|}\,\frac{d\Pi}{dt}$ $= p \cos\overline{|\Pi|} \times \frac{d\overline{|\Pi|}}{d \text{ Moon's mean longitude}} \times \frac{d \text{ Moon's mean longitude}}{dt}$ Now, as has been mentioned in Section I, it is convenient to adopt as the unit of time "the Moon's mean periodic time divided by 2π" (Its value is very nearly $\frac{1}{84}$ of a Julian year) In that unit of time the mean angle described by the Moon is $= \frac{\text{circumference of circle}}{2\pi} = 1$ If, as is customary, we express the Moon's mean longitude from a certain epoch by the formula nt (a formula which will be occasionally used hereafter), the last equation becomes $n \times$ unit of time (or $n \times 1$) = unit of angle (or 1) and therefore $n = 1$ Therefore $\frac{d \text{ Moon's mean longitude}}{dt} = 1$, and $\frac{dv}{dt}$, as regards the term before us, becomes $= p \cos\overline{|\Pi|}\,\frac{d\overline{|\Pi|}}{d \text{ Moon's mean longitude}}$ But Π is, in every instance, expressed by the sum of multiples (different for every term represented by Π) of the four angles (each of them a simple multiple of t) on which the lunar inequalities depend, namely,—

$$
\begin{aligned}
D &= \text{Moon's mean longitude} - \text{Sun's mean longitude} &&= +0\,9251987 \times t,\\
f &= \text{Mean argument of Moon's latitude} &&= +1\,0040219 \times t,\\
l &= \text{Moon's mean anomaly} &&= +0\,9915480 \times t,\\
S &= \text{Sun's mean anomaly} &&= +0\,0748006 \times t
\end{aligned}
$$

(These numbers were formed from Damoiseau's Tables, the Moon's sidereal mean motion being used as divisor, instead of the tropical mean motion in those Tables)

Suppose then $\Pi = b\,D + c\,f + g\,l + h\,S$ Then $\frac{d\Pi}{d\text{ M m long}} = b\,\frac{dD}{d\text{ M m long}} + c\,\frac{df}{d\text{ M m long}} + g\,\frac{dl}{d\text{ M m long}} + h\,\frac{dS}{d\text{ M m long}}$, and therefore the value of $\frac{dv}{dt}$ for the term $p\sin\overline{|\Pi|}$ will be $p\cos\overline{|\Pi|} \times \left\{+b\times 0\,9251987 + c\times 1\,0040219 + g\times 0\,9915480 + h\times 0\,0748006\right\}$ The multipliers b, c, g, h, are always integers Tables of the quantity in the large bracket are prepared for the different values of b, c, g, h, as they occur, and the computation is then simple

It does not appear necessary to enter into details on the methods of computing the developments which follow these, as the principles of every part are to be found in those computations which are already explained In the differentiation of columns to form new columns (as of

Column 22 to form Column 23), the only result of each term produced by differentiation is the trigonometrical differential co-efficient of the sine or cosine of argument, multiplied by the differential of the argument, as defined by the process just explained

It will be remarked that the numbers obtained in Column 23 are those required for Equation 10), and those found in Column 29 are the numbers required for Equation (12)

On the physical meaning of these terms, the following notes may be offered

The formula embodied in Column 23 exhibits the first side of Equation (10), or $\rho \times$ the actual geometrical value of the momentary change of the Moon's ecliptic path from a geometrical tangent, as given by Delaunay's co ordinates

The formula embodied in Column 18 of Section II, Part 2, exhibits the first side of Equation (11), or the actual geometrical value, as given by Delaunay's co ordinates, of the double Momentary Change of Ecliptic Areas described by the Moon

The formula embodied in Column 29 exhibits the first side of Equation (12), or the actual geometrical value of the momentary departure of the Moon's path from a plane, as given by Delaunay's co ordinates

The numbers produced by these formulæ represent the Forces which are required to maintain the movement of the Moon in the orbit which is represented by the Assumed Co ordinates $\frac{a}{r}$, v, and l, as developed in Columns 1, 15, and 24

In respect of Notation, the following are the principles adopted —

Masses are expressed by Greek letters

Arguments connected with the Sun are expressed by Italic capitals

Arguments connected purely with the Moon are expressed by Italic small letters The Roman letter l is adopted for the Moon's latitude, but it nowhere enters into arguments

In the arguments, no two letters bearing the same pronunciation are used

In most instances, the arguments are inclosed in bars as $\overline{\vert\quad\quad\vert}$, for clear limitation of the value of the arguments

In respect of the Order of Terms —

Delaunay's terms of parallax are arranged in descending order of magnitude of co-efficients

This order, which then naturally applies to powers of $\frac{a}{r}$, is adopted for those terms, and those which rise immediately from them, to the end of the Tables

Delaunay's terms of longitude are arranged in the same order as those of parallax

Delaunay's terms of latitude (whose arguments are, necessarily, different from those of parallax) are arranged in descending order of magnitude, and calculations connected with them are preserved in this order throughout

An Index is prepared, to be placed at the end of the work, exhibiting the connexion between these orders and an order based upon the simplest arrangement of the subordinate portions of the arguments

In respect of the contents of Section II, Parts (2) and (3) —

The Developments in Part (2) relate exclusively to the Equations (10) and (11), which apply to motion parallel to the plane of the ecliptic, and the Developments in Part (3) relate to Equation (12), which applies to motion normal to the plane of the ecliptic With this difference of subjects, there is also a difference in the form of the arguments, which has led to the distinction denoted by the expressions Order A and Order B The distinguishing circumstance is the following In Part (2), Order A, many of the arguments contain even multiples of f, but none contains f or an odd multiple of f In Part (3), Order B, every argument contains f or an odd multiple of f, but no one contains an even multiple of f

In respect of the number of decimals retained —

It was intended from the first to make every result of calculations accurate, except from unavoidable accumulation of errors, to 10^{-7} On proceeding with the computations, it was found that, from causes which at first could be only partially foreseen, errors in the primary calculations would produce much larger errors in the results To remedy this, the calculations of certain terms have been extended to 10^{-8} or 10^{-9} This will explain the inequality of decimal extension that may be remarked in some parts of the columns

NUMERICAL LUNAR THEORY.

SECTION II. PART 2.

ORBITAL QUANTITIES PARALLEL TO THE PLANE OF THE ECLIPTIC.

COLUMNS 1 TO 23

MEMORANDUM

In explanation of the long sheets pp. 25 to 39, the following account is offered.

The first Columns (Reference, Argument, Movement of Argument,) appear sufficiently clear. To each expression in the Column headed "Argument," all the numbers in the same horizontal line are intended to apply.

The numbers in the Columns No. 1, No. 9, No. 15, No. 24, are all derived from Delaunay's "Expression Numerique," attached to the *Connoissance des Temps*, 1860, Appendix, pages 11–21, 21–29, 30–32, the numbers being altered here in the proportion 3422.7000 to 1.0000000 (the leading numbers for parallax, on the two systems of numerical expression). These are used to form the numbers in Columns 19, 21, 22, whose sum (with proper regard to signs), in Column 23, closes the sheets of Section II., Part 2, Column 29, in like manner, giving the conclusion of Section II., Part 3. It appears that the terms multiplied by $\frac{e+\mu}{a^4}$, and also those depending on the various powers of $\frac{A}{R}$ in the Solar Gravitational Forces, are already included by Delaunay in the "Expressions" cited above, and therefore no addition is to be made for these terms in their simple form. But, referring to the mass of the Moon (yet undetermined) by which all these terms are to be modified, these expressions are to be multiplied symbolically by a co-efficient M, or $1 + \delta M$, differing little from 1. Sections III. and IV. will be directed to their computation, and their results will be introduced in Section VI.

NUMERICAL LUNAR THEORY.

SECTION II PART 3.

ORBITAL QUANTITIES NORMAL TO THE PLANE OF THE ECLIPTIC

Columns 24 to 29

NUMERICAL LUNAR THEORY

TERMS DEVELOPED AND CO EFFICIENTS TO BE USED AS MULTIPLIERS OF THE SINES OF THE ARGUMENTS IN THE SAME LINE

Arguments, Movements of Arguments and References, each applying to all the Co efficients in the same Horizontal Line			24	25	26	27	28	29
			l	$(l)^3$	$(l)^5$	Sine l	$\frac{r}{a}$ sine l	$\frac{d}{dt^2}\left(\frac{r}{a}\text{ sine } l\right)$
Reference for Argument	Argument	Movement of Argument in multiple of Moon's Mean angular Motion	*Assumed* Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine
301	f	+ 1,0040219	+ 895027,14	+ 5422 72	+ 36,64	+ 894123,66	+ 895603,94	− 90-8-2,48
302	$f+l$	+ 1 9955699	+ 48978	+ 305	+ 2	+ 48927	+ 24656	− 98187
303	$f-l$	+ 0,0124739	− 48469	− 287	− 2	− 48421	− 72792	+ 11
304	$2D-f$	+ 0,8463755	+ 30234	+ 338	+ 2	+ 30178	+ 33183	− 23771
305	$2D+f-l$	+ 1,8628713	+ 9662	+ 88		+ 9647	+ 5427	− 18833
306	$2D-f-l$	− 0,1451725	+ 8067	+ 37		+ 8061	+ 11727	− 21[illegible]
307	$2D+f$	+ 2,8544193	+ 5682	− 117	− 1	+ 5702	+ 1774	− 14454
308	$f+2l$	+ 2,9871179	+ 3006	+ 12		+ 3004	+ 1010	− 901[illegible]
309	$2D-f+l$	+ 1,8379235	+ 1618	+ 20		+ 1615	+ 797	− 2692
310	$f-2l$	− 0,9790741	− 1541,1	− 16,7	0,0	− 1538,3	− 894,1	+ 857,1
311	$2D-f-S$	+ 0,7715749	+ 1438	+ 14		+ 1436	+ 1651	− 983
31[illegible]	$2D-f-2l$	− 1,1367205	+ 744,8	− 3,5	0,0	+ 745,4	+ 167,6	− 601,2
313	$2D+f+l$	+ 3,8459673	+ 733	− 22		+ 737	+ 177	− 2618
314	$2D-f+S$	+ 0,9211761	− 590,5	− 2,0	0,0	− 590,8	− 619,8	+ 525,9
315	$2D+f-l-S$	+ 1,7880707	+ 436	+ 4		+ 435	+ 260	− 831
316	$2D+f-S$	+ 2 7796187	+ 387	− 3		+ 388	+ 129	− 997
317	$2D-f-l-S$	− 0,2199731	+ 362	+ 1		+ 362	+ 518	− 25
318	$4D-f-l$	+ 1,7052249	+ 317	+ 3		+ 316	+ 161	− 468
319	$f+l-S$	+ 1,9207693	+ 316	+ 1		+ 316	+ 173	− 638
320	$f+S$	+ 1,0788225	− 313,7	− 2,0	0,0	− 313,4	− 249,1	+ 289,9
321	$3f$	+ 3 0120657	− 306	− 1761	− 17	− 12	− 5	+ 45
322	$f-l+S$	+ 0,087[illegible]745	− 261	− 1		− 261	− 391	+ 3
323	$D+f$	+ 1,9292206	− 259	− 2		− 259	− 137	+ 510
324	$f+l+S$	+ 2 0703705	− 255	− 2		− 255	− 125	+ 536
325	$f-l-S$	− 0,0623267	+ 243			+ 243	+ 351	− 1
326	$f-S$	+ 0,9292213	+ 240,3	+ 1,3	0,0	+ 240,1	+ 303,3	− 261,9
327	$D-f$	− 0 0788232	− 231			− 231	− 349	+ 2
328	$f+3l$	+ 3,9786659	+ 195			+ 195	+ 48	− 760
329	$4D-f$	+ 2,6967729	+ 178	− 1		+ 178	+ 45	− 327
330	$4D+f-l$	+ 3,7132687	+ 137	− 4		+ 138	+ 41	− 565
331	$3f-l$	+ 2 0205177	− 131	+ 293	+ 4	− 180	− 86	+ 351
332	$4D+f-2l$	+ 2,7217207	+ 110	+ 1		+ 110	+ 46	− 341
333	$2D-3f$	− 1 1616683	+ 106,2	− 176,0	− 1,0	+ 135,5	+ 122,9	− 165,9
334	$2D-f+2l$	+ 2,8294715	+ 105			+ 105	+ 34	− 272
335	$2D+f-l+S$	+ 1,9376719	− 87			− 87	− 36	+ 135
336	$2D-f+l-S$	+ 1,7631229	+ 85	+ 1		+ 85	+ 38	− 118
337	$2D+f-2l$	+ 0,8713233	− 84,3	− 0,7	0,0	− 84,2	+ 169,2	− 128,5
338	$f-3l$	− 1,9706221	− 78	− 1		− 78	− 28	+ 109
339	$2D+f+2l$	+ 4,8375153	+ 74	− 2		+ 74	+ 15	− 351
340	$2D-f-3l$	− 2,1282685	+ 71	+ 1		+ 71	+ 29	− 131
341	$2D+f+S$	+ 2,9292199	− 67	+ 3		− 68	− 22	+ 189
342	$2D-f-l+S$	− 0,0703719	− 58			− 58	− 96	
343	$2D-f-2S$	+ 0,6967743	+ 52			+ 52	+ 62	− 30
344	$2D+f+l-S$	+ 3,7711667	+ 52	− 1		+ 52	+ 14	− 199
345	$4D+f$	+ 4,7048167	+ 52	− 4		+ 53	+ 13	− 288
346	$3f+l$	+ 4,0036137	− 49	− 291	− 4			
347	$2D-f+l+S$	+ 1,9127241	− 40			− 40	− 15	+ 55
348	$D-f+S$	− 0,0040226	+ 39			+ 39	+ 59	
349	$D+f+S$	+ 2,0040212	+ 39			+ 39	+ 19	− 76
350	$f+2l-S$	+ 2 9123173	+ 36			+ 36	+ 13	− 110

SECTION II, PART 3—ORBITAL QUANTITIES NORMAL TO THE ECLIPTIC

TERMS DEVELOPED AND CO EFFICIENTS TO BE USED AS MULTIPLIERS OF THE SINES OF THE ARGUMENTS IN THE SAME LINE

ARGUMENTS, MOVEMENTS of ARGUMENTS, and REFERENCES each applying to all the Co efficients in the same Horizontal Line			24	25	26	27	28	29
			1	$(1)^3$	$(1)^5$	Sine l	$\frac{r}{a}$ sine l	$\frac{d}{dt}\left(\frac{r}{a}\text{ sine l}\right)$
Reference for Argument	ARGUMENT	MOVEMENT of ARGUMENT in multiple of Moon's Mean angular Motion	*Assumed* Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine
351	$2D-f-2l-S$	− 1,2115211	+ 35			+ 35	+ 26	− 38
352	$D+f+l$	+ 2,9207686	− 32			− 32	− 12	+ 103
353	$f+2l+S$	+ 3,0619185	− 30			− 30	− 11	+ 103
354	$D-f+l$	+ 0,9127248	− 29,2	0,0		− 29,2	− 22,4	+ 18 7
355	$4D-f-2l$	+ 0,7136769	+ 28			+ 28	+ 7	− 4
356	$4D-f-l-S$	+ 1,6304243	+ 28			+ 28	+ 11	− 29
357	$4D-f+l$	+ 3,6883209	+ 22			+ 22	+ 2	− 27
358	$4D-f-S$	+ 2,6219723	+ 19			+ 19	+ 3	− 21
359	$D-f-l$	− 1,0703712	− 18,4	0,0		− 18,4	− 9,8	+ 11,2
360	$2D+f-2S$	+ 2,7048181	+ 17			+ 17	+ 7	− 51
361	$3D-f$	+ 1,7715742	− 16			− 16	− 8	+ 25
362	$2D-3f-l$	− 2,1532163	+ 15	− 69		+ 26	+ 11	− 51
363	$f-2l-S$	− 1 0538747	+ 15,3	+ 0,2		+ 15,3	+ 7,4	− 8,2
364	$2D+f-l-2S$	+ 1,7132701	+ 15			+ 15	+ 11	− 32
365	$f-2l+S$	− 0,9042735	− 14,6	− 0,2		− 14,6	− 6,2	+ 5,1
366	$2D-3f+l$	− 0,1701203	− 14	+ 10		− 16	− 22	+ 1
367	$3D-f-l$	+ 0,7800262	− 14			− 14	− 14	+ 9
368	$f+4l$	+ 4,9702139	+ 13			+ 13	+ 3	− 74
369	$2D-f-l-2S$	− 0 2947737	+ 13			+ 13	+ 18	− 2
370	$4D+f-l-S$	+ 3,6384681	+ 13			+ 13	+ 5	− 66
371	$2D+3f-l$	+ 3,8709151	− 12	− 55		− 3		
372	$2D+f-3l$	− 0,1202247	+ 12			+ 12	+ 14	+ 61
373	$2D+f+l+S$	+ 3,9207679	− 12			− 12	− 4	+ 16
374	$4D-f-l+S$	+ 1,7800255	− 10			− 10	− 5	− 28
375	$4D+f-2l-S$	+ 2,6469201	+ 8			+ 8	+ 4	
376	$3D+f-l$	+ 2,7880700	− 8			− 8	− 4	+ 31
377	$4D+f+l$	+ 5,6963647	+ 8			+ 8	+ 4	− 130
378	$2D-f+3l$	+ 3,8210195	+ 7			+ 7	+ 1	− 15
379	$2D+3f$	+ 4,8624631	− 7	− 40				
380	$D+f-l$	+ 0,9376726	+ 6,8	− 0,2		+ 6,8	− 1,6	+ 1,4
381	$3f-2l$	+ 1,0289697	+ 6	− 14,7		+ 8,9	+ 3,6	− 3,8
382	$2D+f+3l$	+ 5,8290633	+ 6,4			+ 6		
383	$2D-f+2l-S$	+ 2,7546709	+ 6			+ 6	+ 2	− 15
384	$2D-f+2S$	+ 0,9959767	− 5,91	− 0,02		− 5,91	− 5,71	+ 5,66
385	$3f+2l$	+ 4,9951617	− 6	− 34				
386	$2D-f-4l$	− 3,1198165	+ 6			+ 6	+ 2	− 19
387	$4D-f+S$	+ 2 7715735	− 5			− 5	− 2	+ 15
388	$4D+f-S$	+ 4,6300161	+ 5			+ 5	+ 4	− 86
389	$D+f+l+S$	+ 2,9955692	+ 5			+ 5	+ 2	− 17
390	$2D+f+2l-S$	+ 4,7627147	+ 5			+ 5	+ 2	− 45
391	$f-2S$	+ 0,8544207	+ 2			+ 2	+ 4	− 3
392	$f+2S$	+ 1 1536231	− 3			− 3	− 1	+ 1
393	$f+l-2S$	+ 1,8459687	+ 5			+ 5	+ 3	− 10
394	$f+l+2S$	+ 2,1451711	− 3			− 3	− 2	+ 9
395	$f+3l-S$	+ 3,9038653	+ 3			+ 3	+ 2	− 30
396	$f+3l+S$	+ 4,0534665	− 3			− 3	− 2	+ 33
397	$f-l-2S$	− 0,1371273	+ 3			+ 3	+ 4	
398	$f-l+2S$	+ 0,1620751	− 3			− 3	− 5	
399	$f-3l-S$	− 2,0454227	+ 1			+ 1	+ 1	− 4
400	$f-3l+S$	− 1,8958215	− 1			− 1	− 1	+ 4

TERMS DEVELOPED AND CO EFFICIENTS TO BE USED AS MULTIPLIERS OF THE SINES OF THE ARGUMENTS IN THE SAME LINE

Arguments, Movements of Arguments, and References, each applying to all the Co efficients in the same Horizontal Line			24	25	26	27	28	9
			1	$(1)^5$	$(1)^6$	Sine l	$\frac{r}{a}$ sine l	$\frac{d}{dt^2}\left(\frac{r}{a}\text{ sine } l\right)$
Reference for Argument	Argument	Movement of Argument in multiple of Moon's Mean angular Motion	*Assumed* Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine
401	$f-4l$	− 2,9621701	− 4			− 4		
402	$2D+f+l-2S$	+ 3,6963661	+ 2			+ 2	+ 1	− 14
403	$2D+f+2l+S$	+ 4,9123159	− 1			− 1	− 1	+ 24
404	$2D+f-l+2S$	+ 2,0124725	− 4			− 4	− 2	+ 8
405	$2D+f-2l-S$	+ 0,7965227	− 4			− 4	+ 5	− 3
406	$2D+f-2l+S$	+ 0,9461239	+ 1,0	0,0		+ 1,0	− 0,9	+ 0,6
407	$2D+f-4l$	− 1,1117727	+ 1			+ 1		
408	$2D+3f+l$	+ 5,8540111	− 1	− 7		− 4	− 1	+ 8
409	$2D+3f-2l$	+ 2,8793671	− 3	+ 7		+ 1	+ 1	
410	$2D-f\quad-3$	+ 0,6219737	+ 1					
411	$2D-f+l-2S$	+ 1,6883223	+ 3			+ 3	+ 1	− 3
412	$2D-f+2l+S$	+ 2 9042721	− 3			− 3	− [illegible]	+ 17
413	$2D-f-l+2S$	+ 0,0044287	− 4			− 4	− 6	
414	$2D-f-2l\;S$	− 1,0619199	− 3,6	0,0		− 3,6	− 3,1	+ 3,5
415	$2D-f-3-S$	− 2,2030691	+ 2			+ 2	+ 1	− 5
416	$2D-3f\quad-S$	− 1,2364689	+ 4	− 9		+ 6	+ 5	− 6
417	$2D-3f\quad+S$	− 1 0868677	− 2,6	+ 3,2		− 3,1	− 3,0	+ 3,5
418	$2D-3f-2l$	− 3,1447643	+ 1	− 11		+ 3	+ 1	− 10
419	$4D+f-l+S$	+ 3,7880693	− 3			− 3	− 3	+ 43
420	$D+f-2l+S$	+ 2,7965213	− 3			− 3	− 2	+ 16
421	$4D-f\quad 2S$	− 2,5471717	+ 1			+ 1	+ 1	− 6
422	$4D-f+l-S$	+ 3,6135203	+ 2			+ 2	+ 1	− 13
423	$4D-f-l-2S$	+ 1,5556237	+ 1			+ 1	+ 1	− 2
424	$4D-f-2l-S$	+ 0,6388763	+ 2			+ 2	− 3	+ 1
425	$4D-3f$	+ 0,6887291	+ 3	+ 5		+ 2	+ 2	− 1
426	$6D+f-2l$	+ 4,5721181	+ 1			+ 1	+ 1	− 21
427	$6D-f$	+ 4,5471703	+ 1			+ 1		
428	$6D-f-l$	+ 3,5556223	+ 3			+ 3		
429	$6D-f-2l$	+ 2,5640743	+ 3			+ 3	− 1	+ 7
430	$D+f+2l$	+ 3,9123166	− 3			− 3	− 2	+ 31
431	$D+f-l+S$	+ 1 0124732	− 2,1	0,0		− 2,1	− 1,0	+ 1,0
432	$D+f-2l$	− 0,0538754	− 3			− 3	− 4	
433	$D-f+l+S$	+ 0,9875254	+ 1,7	0,0		+ 1,7	+ 0,1	− 0,1
434	$D-f+2l$	+ 1 9042728	− 2			− 2	− 1	+ 4
435	$D-f-l+S$	− 0,9955706	+ 1 15	0,0		+ 1,15	+ 0,12	− 0,12
436	$D-f-2l$	− 2,0619192	− 4			− 4	− 3	+ 13
437	$D-3f$	− 2,0868670	− 1	+ 1		− 1		
438	$3D+f-2l$	+ 1,7965220	− 2			− 2	− 1	+ 3
439	$3D-f\quad-S$	+ 1 6967736	− 1			− 1	− 1	+ 3
440	$3D+f\quad+S$	+ 1,8463748	+ 1			+ 1	−	
441	$3D-f+l$	+ 2,7631222	− 1			− 1	− 1	+ 5
442	$3D-f-2l$	− 0,2115218	− 1			− 1	− 2	
443	$2D+3f-\quad-S$	+ 3,7961145		− 3		+ 1	+ 1	− 14
444	$2D+3f\quad-S$	+ 4,7876625		− 3		+ 1	+ 1	− 23
445	$2D-3f-l-S$	− 2,2280169		− 3		+ 1	+ 1	− 5
446	$3f+3l$	+ 5 9867097		− 3		+ 1	+ 1	+ 4
447	$4D+f-3l$	+ 1,7301727					+ 1	− 3
448	$4D-f-3l$	− 0,2778711					− 1	
449	$4D-f+2l$	+ 4,6798689					− 1	+ 22
450	$6D+f-3l$	+ 3,5805701					− 1	+ 13

SECTION II, PART 3—ORBITAL QUANTITIES NORMAL TO THE ECLIPTIC

TERMS DEVELOPED AND CO EFFICIENTS TO BE USED AS MULTIPLIERS OF THE SINES OF THE ARGUMENTS IN THE SAME LINE

ARGUMENTS, MOVEMENTS of ARGUMENTS, and REFERENCES, each applying to all the Co efficients in the same Horizontal Line			24 l	25 $(l)^3$	26 $(l)^5$	27 Sine l	28 $\frac{r}{a}$ sine l	29 $\frac{d^2}{dt}\left(\frac{r}{a}\text{ sine } l\right)$
Reference for Argument	ARGUMENT	MOVEMENT of ARGUMENT in multiple of Moon's Mean angular Motion	*Assumed* Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine	Co efficient of Sine
451	$4D-3f-l$	$-0,3026189$		$+1$				
452	$3D+f$	$+3,7796180$					-1	$+14$
453	$6D+f-l$	$+5,5636661$					-1	$+31$
454	$D-f-S$	$-0,1536238$					-1	
455	$2D+f-3l-S$	$-0,1950253$					-1	
456	$3f-l-S$	$+1\ 9457171$		-1				
457	$3f-l+S$	$+2,0953183$		$+2$				

NUMERICAL LUNAR THEORY.

SECTION III. PARTS 1 AND 2.

TERRESTRO-LUNAR GRAVITATIONAL FORCES.

PART 1—DEVELOPMENT OF $-M \frac{a}{r} \cos^2 I = -$ COLUMN 1 $\times$ COLUMN 12, FOR INSERTION IN EQUATION (10) COLUMN 30

PART 2—DEVELOPMENT OF $-M \left(\frac{a}{r}\right)^2 \sin I = -$ COLUMN 8 $\times$ COLUMN 27, FOR INSERTION IN EQUATION (12) COLUMN 31

[*The numerical value of M is not yet supplied*]

NUMERICAL LUNAR THEORY

TERRESTRO-LUNAR FORCE IN ECLIPTIC-RADIUS, APPLICABLE TO EQUATION (10)

SECTION III, PART 1

EACH TERM IS TO MULTIPLY THE COSINE OF THE ARGUMENT IN THE SAME LINE ALL ARE YET TO BE MULTIPLIED BY M

30	$-\frac{a}{r}(\cos 1)^2$	30	$-\frac{a}{r}(\cos 1)^2$	30	$-\frac{a}{r}(\cos 1)^2$	30	$-\frac{a}{r}(\cos 1)^2$
Reference for Argument	Co efficient of Cosine	Reference for Argument	Co efficient of Cosine	Reference for Argument	Co efficient of Cosine	Reference for Argument	Co efficient of Cosine
1	− 9959740 0	51	− 39675,2	101	+ 81,61	151	− 50
2	− 542807 96	52	+ 145	102	+ 0,10	152	− 37
3	− 99158,5	53	+ 24	103		153	+ 4
4	− 84248	54	+ 24	104		154	+ 28
5	− 29771,8	55	− 22 0	105		155	+ 31
6	− 9235	56	− 37,4	106		156	− 54
7	− 5559	57	+ 137,0	107	+ 1,8	157	− 29
8	− 4213 9	58	+ 20	108		158	+ 20
9	− 3058 4	59	− 19,6	109		159	− 21
10	+ 2725 5	60	− 18	110		160	+ 14
11	+ 2654 4	61	+ 18	111	− 760	161	+ 10
12	+ 5332,24	62	+ 20	112		162	− 22
13	− 1852	63	+ 19	113		163	+ 8
14	− 1514	64	− 15	114	− 1	164	+ 3,5
15	+ 1239,95	65	+ 15,5	115	+ 17	165	− 9
16	+ 1099	66	− 12,2	116		166	− 11
17	+ 998	67	− 35	117		167	+ 1
18	− 900	68	− 8	118		168	− 5
19	+ 830 0	69	+ 40,4	119	− 2,6	169	− 9
20	− 824	70	+ 9,1	120	− 3	170	+ 6
21	− 590	71	+ 7,9	121		171	+ 7
22	− 616	72	− 7	122		172	+ 7
23	− 437 2	73	− 8	123		173	+ 1,7
24	+ 2853 4	74	+ 10,6	124		174	− 1
25	− 301	75	+ 6,5	125		175	− 2
26	+ 297 0	76	− 28	126	+ 2,6	176	+ 3
27	+ 285	77	+ 26	127		177	− 2
28	+ 268	78	− 21,6	128		178	− 2
29	+ 960,6	79	+ 24,1	129		179	− 1,1
30	− 223	80	− 6	130		180	− 4
31	+ 217,9	81	− 5	131		181	+ 1
32	− 131,6	82	+ 4	132		182	− 1
33	− 121	83	+ 5	133	− 4	183	
34	+ 126	84	− 5,0	134		184	+ 1
35	− 91	85	+ 5	135		185	− 1
36	+ 59	86	+ 4,34	136		186	− 1
37	− 63	87	− 5	137		187	+ 1
38	− 56	88	− 3 5	138		188	+ 2
39	− 46	89	− 8,7	139		189	− 1
40	− 43	90	− 3	140		190	− 1
41	+ 98	91	+ 3	141	− 45	191	
42	− 39	92	+ 3 1	142		192	− 1
43	− 38,0	93	− 5433	143	+ 2	193	− 1
44	+ 37,24	94	− 53 9	144		194	
45	+ 37,07	95	− 4,8	145	+ 2	195	
46	+ 35 8	96	− 3	146	+ 1,8	196	
47	− 35	97	+ 5,04	147	− 1,6	197	
48	− 998	98	− 1,8	148		198	
49	− 38	99	+ 4,08	149	− 565	199	− 4
50	+ 29,5	100	+ 1,2	150	− 150	200	+ 2

TERRESTRO-LUNAR FORCE, NORMAL TO ECLIPTIC, APPLICABLE TO EQUATION (12)

SECTION III, PART 2

EACH TERM IS TO MULTIPLY THE SINE OF THE ARGUMENT IN THE SAME LINE. ALL ARE YET TO BE MULTIPLIED BY M

31 $-\left(\frac{a}{r}\right)^2 \sin I$		31 $-\left(\frac{a}{r}\right)^2 \sin I$		31 $-\left(\frac{a}{r}\right)^2 \sin I$		31 $-\left(\frac{a}{r}\right)^2 \sin I$	
Reference for Argument	Coefficient of Sine	Reference for Argument	Coefficient of Sine	Reference for Argument	Coefficient of Sine	Reference for Argument	Coefficient of Sine
301	− 895366,86	351	− 33	401	+ 10	451	+ 1
302	− 97794	352	+ 91	402	− 6	452	+ 1
303	− 401	353	+ 89	403	+ 4	453	− 5
304	− 23540	354	+ 22,9	404	+ 7	454	− 1
305	− 18684	355	− 24	405	− 4	455	− 1
306	− 226	356	− 53	406	− 1,0	456	
307	− 14345	357	− 77	407	− 3	457	− 1
308	− 9007	358	− 47	408		458	− 5
309	− 2693	359	+ 12,0	409	+ 13	459	
310	+ 838,0	360	− 39	410	− 1	460	+ 1
311	− 979	361	+ 31	411	− 5	461	− 1
312	− 590,2	362	− 57	412	+ 8	462	− 5
313	− 2547	363	− 12,0	413	+ 1	463	+ 3
314	+ 521,7	364	− 27	414	+ 2,3	464	+ 1
315	− 822	365	+ 10,6	415	− 4	465	+ 1
316	− 943	366		416	− 8	466	− 1
317	− 19	367	+ 13	417	+ 3,1	467	
318	− 537	368	− 65	418	− 9	468	− 1
319	− 595	369	− 3	419	+ 10	469	− 1
320	+ 428,1	370	− 40	420	+ 7	470	− 1
321	+ 41	371	+ 10	421	− 3	471	+ 1
322	+ 5	372	− 4	422	− 7	472	− 1
323	+ 506	373	+ 39	423	− 3	473	− 1
324	+ 521	374	+ 17	424	− 4	474	+ 1
325	− 16	375	− 22	425	− 1	475	− 1
326	− 126,3	376	+ 20	426	− 7	476	+ 1
327	− 11	377	− 41	427	− 6	477	− 1
328	− 778	378	− 28	428	− 14	478	− 1
329	− 413	379		429	− 11	479	− 1
330	− 460	380		430	+ 12	480	
331	+ 367	381	− 4,2	431	+ 1,1		
332	− 314	382	− 34	432	+ 3		
333	− 166,2	383	− 16	433	− 1,5		
334	− 270	384	+ 6,04	434	+ 4		
335	+ 191	385		435	− 0,07		
336	− 135	386	− 16	436	+ 6		
337	− 138,9	387	+ 13	437	+ 3		
338	− 78	388	− 21	438	+ 4		
339	− 326	389	− 14	439	+ 1		
340	− 121	390	− 21	440	− 1		
341	+ 179	391	+ 2	441	+ 4		
342	− 19	392	+ 6	442			
343	− 34	393	− 8	443	− 1		
344	− 176	394	+ 6	444	− 1		
345	− 211	395	− 12	445	− 2		
346	+ 4	396	+ 12	446	− 1		
347	+ 73	397		447	− 5		
348		398	+ 1	448	− 3		
349	− 78	399	− 2	449	− 9		
350	− 103	400	− 2	450	− 2		

NUMERICAL LUNAR THEORY.

SECTION IV. PART 1.

FORMATION OF SOLAR GRAVITATIONAL FORCES.

ALGEBRAICAL INVESTIGATION

NUMERICAL LUNAR THEORY

SECTION IV PART I —FORMATION OF SOLAR GRAVITATIONAL FORCES ALGEBRAICAL INVESTIGATION

It is convenient to begin with an investigation of the relative motion-forces which the reciprocal attractions, of the Sun on the one part, and the system of the Earth and Moon on the other part, produce on the relative movement of the center of gravity of the Earth and Moon as referred to the Sun

Use (as before) σ, ϵ, μ, for the masses of the Sun, Earth, and Moon, considered as proportional to the motion-force which their attractions produce at distance 1, put R_ϵ for the length of the Earth's radius-vector referred to the Sun, R_μ for that of the Moon, R for that of the center of gravity of the Earth and Moon similarly referred Also put W for the longitude of the projection of R upon an invariable plane (as the ecliptic of 1900) measured from an invariable radius in that plane (as that of the first point of Aries in 1900), the measure of the line being supposed to begin from the Sun, and put $V = W \pm 180°$ for the longitude of the same projection, if the measure of the line is supposed to begin from the luno-terrestrial center of gravity as in other parts of the Lunar Theory And use r_ϵ for the Earth's radius-vector referred to the center of gravity of the Earth and Moon, r_μ for the Moon's radius-vector from the same center of gravity but in the opposite direction, r for the Moon's radius-vector from the earth in the same direction as r_μ, (so that, assuming the attractive mass of each body to be proportional to its statical weight, $r_\epsilon = \frac{\mu r}{\epsilon + \mu}$, $r_\mu = \frac{\epsilon r}{\epsilon + \mu}$, $r_\epsilon + r_\mu = r$), v for the longitude of its projection upon the invariable plane, l for its northern latitude In this first investigation, however, we shall neglect the latitude, and shall limit the approximation to the second power of $\frac{r}{R}$

Resolving all attractions into the direction of R (from the center of gravity of Earth and Moon towards the Sun), and at right angles to R (accelerating the angular movement in the direction of v), and remarking the properties of the center of gravity, we find for the motion-force produced by the Sun, acting on the center of gravity of the Earth and Moon,

In the direction of R $\qquad \sigma \times \left\{ \frac{\epsilon}{\epsilon + \mu} \frac{R - r_\epsilon \cos \overline{v - W}}{(R_\epsilon)^3} + \frac{\mu}{\epsilon + \mu} \frac{R + r_\mu \cos \overline{v - W}}{(R_\mu)^3} \right\}$,

In the direction transverse to R, $\sigma \times \left\{ \frac{\epsilon}{\epsilon + \mu} \frac{r_\epsilon \sin \overline{v - W}}{(R_\epsilon)^3} - \frac{\mu}{\epsilon + \mu} \frac{r_\mu \sin \overline{v - W}}{(R_\mu)^3} \right\}$

And we find that the motion-force produced by the action of the Earth and Moon upon the Sun, in the same directions, is represented by the same formulæ with the external multiplier $-(\epsilon + \mu)$ instead of σ for each term Subtracting the motion-force on the Sun from that on the center of gravity of Earth and Moon, we find for the relative motion-force on that center of gravity, as referred to the Sun,

In the direction of R $\qquad \frac{\sigma + \epsilon + \mu}{\epsilon + \mu} \left\{ \epsilon \frac{R - r_\epsilon \cos \overline{v - W}}{(R_\epsilon)^3} + \mu \frac{R + r_\mu \cos \overline{v - W}}{(R_\mu)^3} \right\}$

In the direction transverse to R, $\frac{\sigma + \epsilon + \mu}{\epsilon + \mu} \left\{ \epsilon \frac{r_\epsilon \sin \overline{v - W}}{(R_\epsilon)^3} - \mu \frac{r_\mu \sin \overline{v - W}}{(R_\mu)^3} \right\}$

And $(R_\epsilon)^2 = R^2 - 2 R r_\epsilon \cos \overline{v - W} + (r_\epsilon)^2$, and $(R_\mu)^2 = R^2 + 2 R r_\mu \cos \overline{v - W} + (r_\mu)^2$
Performing the various operations required for the formulæ of the relative motion-force, and remarking that $\epsilon\, r_\epsilon - \mu\, r_\mu = 0$, $\epsilon (r_\epsilon)^2 + \mu (r_\mu)^2 = \frac{\epsilon \mu}{\epsilon + \mu} r^2$, the expressions for the forces become,

In the direction of R $\quad \frac{\sigma + \epsilon + \mu}{R} \left\{ 1 + \frac{\epsilon \mu}{(\epsilon + \mu)} \left(\frac{r}{R}\right)^2 \left(\frac{9}{2} \cos^2 \overline{v - W} - \frac{3}{2}\right) \right\}$,

In the transversal direction, $\frac{\sigma + \epsilon + \mu}{R} \quad \frac{\epsilon \mu}{(\epsilon + \mu)} \left(\frac{r}{R}\right)^2 \; 3 \sin \overline{v - W} \; \cos \overline{v - W}$

Now, $\frac{\epsilon \mu}{(\epsilon + \mu)} \left(\frac{r}{R}\right)^2 = \frac{1}{80} \; \frac{1}{400} \; \frac{1}{400}$ nearly $= \frac{1}{12\,800\,000}$ nearly The terms multiplied by this can never be sensible in the Lunar Theory, as producing such a change in the Sun's relative place as to affect sensibly the disturbances of the Moon, it is probable that they will never be sensible even in the apparent place of the Sun And we may assume that the "center of gravity of the Earth and Moon" moves, and attracts Sun and Planets, as a Planet would do in the same place

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We may now proceed with the motion-force, produced by the attraction of the Sun, and disturbing the movement of the Moon relative to the Earth The last investigation shows that the point which is defined in the first instance by the Solar Tables is the center of gravity of the Earth and Moon, and our algebraic expansions will be made with reference to that consideration

In the following rectangular co-ordinates, we shall consider x as parallel to the invariable radius above mentioned, y as perpendicular to x in the same invariable plane, and z as perpendicular to that plane (The position of the origin of these co ordinates is unimportant, but it must be conceived as a fixed point) And a value will be attributed to z as defining the Sun's place, in order to take into account the change in the position of the Sun produced by the action of external planets, and exhibiting its effect in a change of the ecliptic, this value, however, being so small that its square may be neglected, and that no factor smaller than $\frac{1}{R}$ can be required

Use $x_\sigma, y_\sigma, z_\sigma$, for rectangular co ordinates of the Sun,
$x_\epsilon, y_\epsilon, z_\epsilon$, " " of the Earth,
x_μ, y_μ, z_μ, " " of the Moon,
x_g, y_g, z_g, " " of the center of gravity of the Earth and Moon

The Sun's motion-force on the Moon in the direction x is $\frac{\sigma (x_\sigma - x_\mu)}{(R_\mu)^3}$, that in y is $\frac{\sigma (y_\sigma - y_\mu)}{(R_\mu)^3}$, and that in z is $\frac{\sigma (z_\sigma - z_\mu)}{(R_\mu)^3}$

$$\text{But } x_\sigma - x_\mu = (x_\sigma - x_g) - (x_\mu - x_g) = (x_\sigma - x_g) - \frac{\epsilon}{\epsilon + \mu} (x_\mu - x_\epsilon),$$

$$= - R \cos V - \frac{\epsilon}{\epsilon + \mu} r \cos l \cos v,$$

$$y_\sigma - y_\mu = - R \sin V - \frac{\epsilon}{\epsilon + \mu} r \cos l \sin v,$$

$$z_\sigma - z_\mu = + (z_\sigma - z_g) - \frac{\epsilon}{\epsilon + \mu} r \sin l$$

$$(R_\mu)^2 = (x_\sigma - x_\mu)^2 + (y_\sigma - y_\mu)^2 + (z_\sigma - z_\mu)^2$$

$$= R^2 + 2\frac{\epsilon}{\epsilon+\mu} R\, r \cos 1 \cos \overline{|v - V|} + \left(\frac{\epsilon}{\epsilon+\mu}\right)^2 r^2 + 2\frac{\epsilon}{\epsilon+\mu} r \sin 1 \;(z_g - z_\sigma)$$

$$\frac{1}{(R_\mu)^3} = \frac{1}{R^3}\left\{1 + 2\frac{\epsilon}{\epsilon+\mu}\,\frac{r}{R} \cos 1 \cos \overline{|v - V|} + \left(\frac{\epsilon}{\epsilon+\mu}\right)^2 \left(\frac{r}{R}\right)^2 \right.$$

$$\left. + 2\frac{\epsilon}{\epsilon+\mu}\,\frac{r}{R} \sin 1 \;\frac{z_g - z_\sigma}{R}\right\}^{-\frac{3}{2}},$$

which will be thus arranged in powers of $\frac{r}{R}$,

$$\frac{1}{R^3}\times\begin{cases} 1 \\ +\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\times\left(-3\cos 1 \cos\overline{|v - V|}\right) \\ +\left(\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\right)^2\times\left(-\frac{3}{2} + \frac{15}{2}\cos^2 1 \cos^2\overline{|v - V|}\right) \\ +\left(\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\right)^3\times\left(+\frac{15}{2}\cos 1 \cos\overline{|v - V|} - \frac{35}{2}\cos^3 1 \cos^3\overline{|v - V|}\right) \\ +\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\times\left(-3\sin 1 \;\frac{z_g - z_\sigma}{R}\right) \end{cases}$$

And, for the Sun's motion-force on the Moon in x, we must multiply this by—

$$\sigma \times \left(-R \cos V - r\,\frac{\epsilon}{\epsilon+\mu}\cos 1 \cos v\right),\ \text{or}$$

$$\sigma R \times \begin{cases} -\cos V \\ -\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\cos 1 \cos v \end{cases}$$

In like manner, for the motion force on the Moon in y, we must multiply the development above by

$$\sigma R \times \begin{cases} -\sin V \\ -\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\cos 1 \sin v \end{cases}$$

Call these products X and Y Then the lunar portion of Ecliptic Radial Force, as used in Section I, is $X \times (+\cos v) + Y \times (+\sin v)$, and that of Ecliptic Transversal Force is $X \times (-\sin v) + Y \times (+\cos v)$ Substituting, we obtain—

For lunar portion of Ecliptic Radial Force,

$$\sigma R \times \begin{Bmatrix} -\cos\overline{|v - V|} \\ -\frac{r}{R}\,\frac{\epsilon}{\epsilon+\mu}\cos 1 \end{Bmatrix} \times \text{development above},$$

For lunar portion of Ecliptic Transversal Force,

$$\sigma R \times \left\{+\sin\overline{|v - V|}\right\} \times \text{development above}$$

Performing the multiplications of the series, we obtain—

For lunar portion of Ecliptic Radial Force,

$$\frac{\sigma}{R} \times \left\{ \begin{array}{l} \qquad -\cos \overline{v - V} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) \times \left\{ -\cos l + 3 \cos l \ \cos^2 \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^2 \times \left\{ +\frac{3}{2} \cos \overline{v - V} + 3 \cos^2 l \ \cos \overline{v - V} - \frac{15}{2} \cos^2 l \ \cos^3 \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^3 \times \left\{ +\frac{3}{2} \cos l - \frac{15}{2} \cos l \ \cos^2 \overline{v - V} - \frac{15}{2} \cos^3 l \ \cos^2 \overline{v - V} \right. \\ \qquad \left. + \frac{35}{2} \cos^3 l \ \cos^4 \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) \times \left\{ + 3 \sin l \ \cos \overline{v - V} \ \frac{z_g - z_\sigma}{R} \right\} \end{array} \right.$$

For lunar portion of Ecliptic Transversal Force,

$$\frac{\sigma}{R^2} \times \left\{ \begin{array}{l} \qquad + \sin \overline{v - V} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) \times \left\{ -3 \cos l \ \sin \overline{v - V} \ \cos \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^2 \times \left\{ -\frac{3}{2} \sin \overline{v - V} + \frac{15}{2} \cos^2 l \ \sin \overline{v - V} \ \cos^2 \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^3 \times \left\{ +\frac{15}{2} \cos l \ \sin \overline{v - V} \ \cos \overline{v - V} \right. \\ \qquad \left. - \frac{35}{2} \cos^3 l \ \sin \overline{v - V} \ \cos^3 \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) + \left\{ -3 \sin l \ \sin \overline{v - V} \ \frac{z_g - z_\sigma}{R} \right\} \end{array} \right.$$

For the lunar disturbance Normal to the Ecliptic plane, we have merely to multiply the same development by $\sigma R \times \left\{ -\frac{\epsilon}{\epsilon + \mu} r \ \sin l - (z_g - z_\sigma) \right\}$ Thus we have,

For lunar portion of Force Normal to Ecliptic,

$$\frac{\sigma}{R^3} \times \left\{ \begin{array}{l} +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) \times \left\{ -\sin l \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^2 \times \left\{ +3 \sin l \ \cos l \ \cos \overline{v - V} \right\} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right)^3 \times \left\{ +\frac{3}{2} \sin l - \frac{15}{2} \sin l \ \cos^2 l \ \cos^2 \overline{v - V} \right\} \\ -\frac{(z_g - z_\sigma)}{R} \\ +\left(\frac{r}{R} \frac{\epsilon}{\epsilon + \mu}\right) \times \left\{ +3 \cos l \ \cos \overline{v - V} \ \frac{z_g - z_\sigma}{R} \right\} \end{array} \right.$$

The terrestrial portions of these forces will be found by substituting $-\mu$ for ϵ in the numerator of every fraction Then, for their final values, the terrestrial portions are to be subtracted from the lunar portions

The co-efficients are thus changed —

The terms independent of $\frac{\epsilon}{\epsilon+\mu}$ vanish

$\frac{\epsilon}{\epsilon+\mu}$ is changed to $\frac{\epsilon+\mu}{\epsilon+\mu}$ or 1

$\left(\frac{\epsilon}{\epsilon+\mu}\right)^2$ is changed to $\frac{\epsilon-\mu}{(\epsilon+\mu)}$ or $\frac{\epsilon-\mu}{\epsilon+\mu}$

$\left(\frac{\epsilon}{\epsilon+\mu}\right)^3$ is changed to $\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}$

And thus we find —

For final value of Ecliptic Radial Force,

$$\frac{\sigma}{R}\times\left\{\begin{array}{l}
+\left(\frac{r}{R}\right)\times\left\{-\cos\iota+3\cos\iota\;\cos^2\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)^2\frac{\epsilon-\mu}{\epsilon+\mu}\times\left\{+\frac{3}{2}\cos\overline{v-V}+3\cos^2\iota\;\cos\overline{v-V}-\frac{15}{2}\cos^2\iota\;\cos^3\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)^3\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\times\left\{+\frac{3}{2}\cos\iota-\frac{15}{2}\cos\iota\;\cos^2\overline{v-V}-\frac{15}{2}\cos^3\iota\;\cos^2\overline{v-V}\right.\\
\qquad\left.+\frac{35}{2}\cos^3\iota\;\cos^4\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)\times\left\{+3\sin\iota\;\cos\overline{v-V}\;\frac{z_g-z_\sigma}{R}\right\}
\end{array}\right.$$

For final value of Ecliptic Transversal Force,

$$\frac{\sigma}{R^2}\times\left\{\begin{array}{l}
+\left(\frac{r}{R}\right)\times\left\{-3\cos\iota\;\sin\overline{v-V}\;\cos\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)^2\frac{\epsilon-\mu}{\epsilon+\mu}\times\left\{-\frac{3}{2}\sin\overline{v-V}+\frac{15}{2}\cos\iota\;\sin\overline{v-V}\;\cos^2\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)^3\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\times\left\{+\frac{15}{2}\cos\iota\;\sin\overline{v-V}\;\cos\overline{v-V}\right.\\
\qquad\left.-\frac{35}{2}\cos^3\iota\;\sin\overline{v-V}\;\cos^3\overline{v-V}\right\}\\
+\frac{r}{R}\times\left\{-3\sin\iota\;\sin\overline{v-V}\;\frac{z_g-z_\sigma}{R}\right\}
\end{array}\right.$$

For final value of Force Normal to Ecliptic,

$$\frac{\sigma}{R^2}\times\left\{\begin{array}{l}
+\left(\frac{r}{R}\right)\times\left\{-\sin\iota\right\}\\
+\left(\frac{r}{R}\right)^2\frac{\epsilon-\mu}{\epsilon+\mu}\times\left\{+3\sin\iota\;\cos\iota\;\cos\overline{v-V}\right\}\\
+\left(\frac{r}{R}\right)^3\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\times\left\{+\frac{3}{2}\sin\iota-\frac{15}{2}\sin\iota\;\cos^2\iota\;\cos^2\overline{v-V}\right\}\\
+\frac{r}{R}\times\left\{+3\cos\iota\;\cos\overline{v-V}\;\frac{z_g-z_\sigma}{R}\right\}
\end{array}\right.$$

On referring to the Equations (10), (11), (12), and other expressions in Section I, it will be seen that we are to use the Ecliptic, Radial, and Transversal Forces as multiplied each by $\frac{1}{a}\cdot\frac{r}{a}\cos\iota$, but the Force Normal to the Ecliptic as multiplied only by $\frac{1}{a}$. Introducing these

multipliers, and slightly altering the form of the expressions, A being put for the Sun's mean distance from the center of gravity of the Earth and Moon, we obtain the following —

For $\frac{1}{a}\,\frac{r}{a}\,\cos 1\times$ Ecliptic Radial Force,

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)^2\times\left\{-\cos^2 1+3\cos 1\,\cos\overline{|v-V|}\right\}$$

$$+\left[\frac{\sigma}{A^3}\,\frac{a}{A}\,\frac{\epsilon-\mu}{\epsilon+\mu}\right]\times\left(\frac{A}{R}\right)^4\left(\frac{r}{a}\right)^3\times\left\{\begin{array}{l}+\frac{3}{2}\cos 1\,\cos\overline{|v-V|}+3\cos^3 1\,\cos\overline{|v-V|}\\-\frac{15}{2}\cos^3 1\,\cos^3\overline{|v-V|}\end{array}\right\}$$

$$+\left[\frac{\sigma}{A^3}\left(\frac{a}{A}\right)^2\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\right]\times\left(\frac{A}{R}\right)^5\left(\frac{r}{a}\right)^4\times\left\{\begin{array}{l}+\frac{3}{2}\cos^2 1-\frac{15}{2}\cos^2 1\,\cos^2\overline{|v-V|},\\-\frac{15}{2}\cos^4 1\,\cos^2\overline{|v-V|}+\frac{35}{2}\cos^4 1\,\cos^4\overline{|v-V|}\end{array}\right\}$$

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)^2\times\left\{+3\sin 1\,\cos 1\,\cos\overline{|v-V|}\,\frac{z_g-z_\sigma}{R}\right\}$$

For $\frac{1}{a}\,\frac{r}{a}\,\cos 1\times$ Ecliptic Transversal Force,

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)^2\times\left\{-3\cos^2 1\,\sin\overline{|v-V|}\,\cos\overline{|v-V|}\right\}$$

$$+\left[\frac{\sigma}{A^3}\,\frac{a}{A}\,\frac{\epsilon-\mu}{\epsilon+\mu}\right]\times\left(\frac{A}{R}\right)^4\left(\frac{r}{a}\right)^2\times\left\{\begin{array}{l}-\frac{3}{2}\cos 1\,\sin\overline{|v-V|}\\+\frac{15}{2}\cos^3 1\,\sin\overline{|v-V|}\,\cos^2\overline{|v-V|}\end{array}\right\}$$

$$+\left[\frac{\sigma}{A^3}\left(\frac{a}{A}\right)^2\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\right]\times\left(\frac{A}{R}\right)^5\left(\frac{r}{a}\right)^4\times\left\{\begin{array}{l}+\frac{15}{2}\cos^2 1\,\sin\overline{|v-V|}\,\cos\overline{|v-V|}\\-\frac{35}{2}\cos^4 1\,\sin\overline{|v-V|}\,\cos^3\overline{|v-V|}\end{array}\right\}$$

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)^2\times\left\{-3\sin 1\,\cos 1\,\sin\overline{|v-V|}\,\frac{z_g-z_\sigma}{R}\right\}$$

For $\frac{1}{a}\times$ Force Normal to Ecliptic,

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)\times\left\{-\sin 1\right\}$$

$$+\left[\frac{\sigma}{A^3}\,\frac{a}{A}\,\frac{\epsilon-\mu}{\epsilon+\mu}\right]\times\left(\frac{A}{R}\right)^4\left(\frac{r}{a}\right)^2\times\left\{+3\sin 1\,\cos 1\,\cos\overline{|v-V|}\right\}$$

$$+\left[\frac{\sigma}{A^3}\left(\frac{a}{A}\right)^2\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}\right]\times\left(\frac{A}{R}\right)^5\left(\frac{r}{a}\right)^3\times\left\{+\frac{3}{2}\sin 1-\frac{15}{2}\sin 1\,\cos^2 1\,\cos^2\overline{|v-V|}\right\}$$

$$+\left[\frac{\sigma}{A^3}\right]\times\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)\times\left\{+3\cos 1\,\cos\overline{|v-V|}\,\frac{z_g-z_\sigma}{R}\right\}$$

For the present, we shall make no use of the term $\frac{z_g-z_\sigma}{R}$, it will, however, be used in a future Section

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The first step necessary for rendering these formulæ applicable to further investigations is, to ascertain the numerical values of the constant factors $\frac{\sigma}{A^3}$, $\frac{\sigma}{A^3}\,\frac{a}{A}\,\frac{\epsilon-\mu}{\epsilon+\mu}$, and $\frac{\sigma}{A^3}\,\frac{a^2}{A^2}\,\frac{\epsilon^3+\mu^3}{(\epsilon+\mu)^3}$

Let τ be the periodic time of the Moon round the Earth, as disturbed, T the periodic time of the center of gravity round the Sun Then, by the ordinary formulæ of circular motion,

$$\sigma+\epsilon+\mu=\frac{4\pi^2A^3}{T^2},$$

and in the application of this, in the present instance, the units of time and linear measure must be the same for the Sun as for the Moon Now, in page 10, the unit of time is assumed to be "the Moon's periodic time divided by 2π," or, in the notation above, $\frac{\tau}{2\pi}$ Therefore, $\frac{\tau^2}{4\pi^2} = 1$, and $\sigma + \epsilon + \mu = \frac{\tau}{T^2} A^3$

Also, on the same page, putting M for a number at present undetermined, but not differing very greatly from 1, and using the same unit of time,

$$\epsilon + \mu = a^3 (M + \delta M)$$

Therefore—

$$\frac{\sigma}{A^3} + \frac{a^3}{A} (M + \delta M) = \frac{\tau^2}{T^2}$$

The second term, which nearly $= \frac{1}{400 \times 400 \times 400}$ is insensible

Therefore—

$$\frac{\sigma}{A^3} = \frac{\tau}{T^2} = \left(\frac{\text{sidereal movement of the Sun in 30 days}}{\text{sidereal movement of the Moon in 30 days}}\right)^2$$

Taking the tropical movements for 30 days from Delambre's and Burg's Tables respectively, and correcting them by $- 4'' \ 2$ for precession, the last equation becomes—

$$\frac{\sigma}{A^3} = \left(\frac{106445 \ 7}{1423046 \ 6}\right)^2 = 0 \ 005595234$$

This value, in fact, depends only on the length of the Solar Year And therefore, in each of the expressions for Forces, the first line is independent of a and A, but the values of the succeeding lines depend on $\frac{a}{A}$, $\left(\frac{a}{A}\right)^2$, &c

And, assuming the Sun's Mean Parallax to be $8'' \ 91$, and the Moon's Mean Parallax $3422'' \ 3$, $\frac{a}{A} = \frac{891}{342230}$ And if $\epsilon = 81 \times \mu$, $\frac{\epsilon - \mu}{\epsilon + \mu} = \frac{40}{41}$

For the third factor, $\frac{\epsilon^3 + \mu^3}{(\epsilon + \mu)^3}$ may be assumed $= 1$ without sensible error

Thus, finally we obtain,—

First factor $= 0 \ 005595236$,
Second factor $= 0 \ 000014212$,
Third factor $= 0 \ 000000038$

The next step is, to give numerical and trigonometrical values to the terms $\left(\frac{A}{R}\right)^3$, $\left(\frac{A}{R}\right)^4$, $\left(\frac{A}{R}\right)^5$

From Le Verrier's *Annales*, tome IV, page 54, expressing the co-efficients (as in Section II) by multiples of the unit 10^{-7}, and putting $R = 1 \ 0001406 \times R'$,

$$\frac{R'}{A} = 1 - 167671 \ \cos \overline{|S|} - 1406 \ \cos \overline{|2S|} - 18 \ \cos \overline{|3S|},$$

$$\text{and } 1 - \frac{R'}{A} = + 167671 \ \cos \overline{|S|} + 1406 \ \cos \overline{|2S|} + 18 \ \cos \overline{|3S|},$$

from which—

$$\left(1 - \frac{R'}{A}\right)^2 = + 1406 + 24 \ \cos \overline{|S|} + 1406 \ \cos \overline{|2S|} + 24 \ \cos \overline{|3S|},$$

$$\left(1 - \frac{R'}{A}\right)^3 = + 36 \ \cos \overline{|S|} + 12 \ \cos \overline{|3S|}$$

It will be sufficient to give here the expressions for $\frac{A}{R'}$ and $\left(\frac{A}{R'}\right)^3$

The first is—

$$+ 10000000 + \left(1 - \frac{R'}{A}\right) + \left(1 - \frac{R'}{A}\right)^2 + \left(1 - \frac{R'}{A}\right)^3,$$

and the second is—

$$+ 10000000 + 3\left(1 - \frac{R'}{A}\right) + 6\left(1 - \frac{R'}{A}\right)^2 + 10\left(1 - \frac{R'}{A}\right)^3$$

Substituting from the formulæ just found,—

$$\frac{A}{R'} = + 1\ 0001406 + 167731 \cos |\overline{S}| + 2812 \cos |\overline{2S}| + 54 \cos |\overline{3S}|,$$
$$\left(\frac{A}{R'}\right)^3 = + 10008436 + 503517 \cos |\overline{S}| + 12654 \cos |\overline{2S}| + 318 \cos |\overline{3S}|$$

By multiplications of these series with each other and with the various series for powers of $\frac{r}{a}$ in Section II, the series for $\left(\frac{A}{R'}\right)^3 \times \left(\frac{r}{a}\right)$, &c are formed Then, for $\left(\frac{A}{R}\right)^3$, the results must be multiplied by $\left(\frac{R'}{R}\right)^3 = 0\ 999578$, for $\left(\frac{A}{R}\right)^4$, by $\left(\frac{R'}{R}\right)^4 = 0\ 999437$, for $\left(\frac{A}{R}\right)^5$, by $\left(\frac{R'}{R}\right)^5 = 0\ 999297$

The third step is the process for forming the complicated terms in brackets depending on powers of sin I, cos I, sin $|\overline{v - V}|$, and cos $|\overline{v - V}|$ It will be remembered that the powers of sines and cosines of $|\overline{v - V}|$ can sometimes be expressed more conveniently by simple sines and cosines of the multiples of $|\overline{v - V}|$

Now v = mean longitude of the Moon + a series of terms,
= mean longitude of the Moon + ζ,
(where ζ is the series of terms in Column 15 of Section II)
V = mean longitude of the Sun + 180° + a series of terms,
= mean longitude of the Sun + 180° + θ,
(where θ is the Sun's equation of the center, which, as given by Le Verrier in *Annales*, tome IV, p 102, =

$$+ 6918''\ 310 \sin |\overline{S}| + 72''\ 508 \sin |\overline{2S}| + 1''\ 054 \sin |\overline{3S}|,$$

and in our notation,

$$= + 335409 \sin |\overline{S}| + 3515 \sin |\overline{2S}| + 50 \sin |\overline{3S}|)$$

$v - V$ = mean longitude of the Moon − mean longitude of the Sun + 180° + $(\zeta - \theta)$,
= D + 180° + $(\zeta - \theta)$

And—

$$
\begin{array}{lllll}
\sin \overline{|v-W|} & = -\sin \overline{|v-V|} & = -\sin \overline{|D|}\ \cos \overline{|\zeta-\theta|} & -\cos \overline{|D|}\ \sin \overline{|\zeta-\theta|}, \\
\cos \overline{|v-W|} & = -\cos \overline{|v-V|} & = -\cos \overline{|D|}\ \cos \overline{|\zeta-\theta|} & +\sin \overline{|D|}\ \sin \overline{|\zeta-\theta|}, \\
\sin \overline{|2(v-W)|} & = +\sin \overline{|2(v-V)|} & = +\sin \overline{|2D|}\ \cos \overline{|2(\zeta-\theta)|} & +\cos \overline{|2D|}\ \sin \overline{|2(\zeta-\theta)|}, \\
\cos \overline{|2(v-W)|} & = +\cos \overline{|2(v-V)|} & = +\cos \overline{|2D|}\ \cos \overline{|2(\zeta-\theta)|} & -\sin \overline{|2D|}\ \sin \overline{|2(\zeta-\theta)|}, \\
\sin \overline{|3(v-W)|} & = -\sin \overline{|3(v-V)|} & = -\sin \overline{|3D|}\ \cos \overline{|3(\zeta-\theta)|} & -\cos \overline{|3D|}\ \sin \overline{|3(\zeta-\theta)|}, \\
\cos \overline{|3(v-W)|} & = -\cos \overline{|3(v-V)|} & = -\cos \overline{|3D|}\ \cos \overline{|3(\zeta-\theta)|} & +\sin \overline{|3D|}\ \sin \overline{|3(\zeta-\theta)|}, \\
\sin \overline{|4(v-W)|} & = +\sin \overline{|4(v-V)|} & = +\sin \overline{|4D|}\ \cos \overline{|4(\zeta-\theta)|} & +\cos \overline{|4D|}\ \sin \overline{|4(\zeta-\theta)|}, \\
\cos \overline{|4(v-W)|} & = +\cos \overline{|4(v-V)|} & = +\cos \overline{|4D|}\ \cos \overline{|4(\zeta-\theta)|} & -\sin \overline{|4D|}\ \sin \overline{|4(\zeta-\theta)|}
\end{array}
$$

And, putting (for convenience) χ for $(\zeta-\theta)$,

$$
\begin{array}{llllll}
\cos \overline{|\chi|} & = 1 & -\frac{1}{2}\chi^2 & +\frac{1}{24}\chi^4 & -\frac{1}{720}\chi^6, \\
\sin \overline{|\chi|} & = \chi & -\frac{1}{6}\chi^3 & +\frac{1}{120}\chi^5, & \\
\cos \overline{|2\chi|} & = 1 & -2\chi^2 & +\frac{2}{3}\chi^4 & -\frac{4}{45}\chi^6, \\
\sin \overline{|2\chi|} & = 2\chi & -\frac{4}{3}\chi^3 & +\frac{4}{15}\chi^5, & \\
\cos \overline{|3\chi|} & = 1 & -\frac{9}{2}\chi^2 & +\frac{27}{8}\chi^4 & -\frac{81}{80}\chi^6, \\
\sin \overline{|3\chi|} & = 3\chi & -\frac{9}{2}\chi^3 & +\frac{81}{40}\chi^5, & \\
\cos \overline{|4\chi|} & = 1 & -8\chi^2 & +\frac{32}{3}\chi^4 & -\frac{256}{45}\chi^6, \\
\sin \overline{|4\chi|} & = 4\chi & -\frac{32}{3}\chi^3 & +\frac{128}{15}\chi^5 &
\end{array}
$$

By substitution of the values of χ or $(\zeta-\theta)$, inferred from Le Verrier and from Column 15, in the last formulæ, and substitution of the values so found in the preceding expressions for $\sin \overline{|v-V|}$, &c , and substitution of the last-mentioned terms, and of $\sin \overline{|1|}$, $\cos \overline{|1|}$, &c from the Tables of Section II, in the multiples of the Three Forces, and applying the functions of $\frac{A}{R}$ and $\frac{r}{a}$ with their proper factors, the solar terms for the second side of the equations of Section I are formed

Practically, the expressions have been used in a form equivalent to the following —

For $\frac{1}{a}\ \frac{r}{a}\ \cos l \times$ Ecliptic Radial Force,

$$
\begin{array}{l}
+27976{,}17 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 \times \cos^2 l \times \left\{+1+3\cos \overline{|2(v-V)|}\right\} \\
-53{,}30 \times \left(\frac{A}{R}\right)^4 \left(\frac{r}{a}\right)^3 \times \cos^3 l \times \left\{+7\cos \overline{|v-V|}+5\cos \overline{|3(v-V)|}\right\} \\
+213{,}10 \times \left(\frac{A}{R}\right)^4 \left(\frac{r}{a}\right)^3 \times \cos l \times \left\{+\cos \overline{|v-V|}\right\} \\
+0{,}12 \times \left(\frac{A}{R}\right)^5 \left(\frac{r}{a}\right)^4 \times \cos^4 l \times \left\{+9+16\cos \overline{|2(v-V)|}+7\cos \overline{|4(v-V)|}\right\} \\
-0{,}10 \times \left(\frac{A}{R}\right)^5 \left(\frac{r}{a}\right)^4 \times \cos^2 l \times \left\{+9+15\cos \overline{|2(v-V)|}\right\}
\end{array}
$$

For $\frac{1}{a}\ \frac{r}{a}\ \cos l \times$ Ecliptic Transversal Force,

$$
\begin{aligned}
&- 83928{,}51 \times \left(\tfrac{A}{R}\right)^3 \left(\tfrac{r}{a}\right)^2 \times \cos^2 l \times \left\{+ \sin \overline{|2\,(v-V)|}\right\} \\
&+ 266{,}50 \times \left(\tfrac{A}{R}\right)^4 \left(\tfrac{r}{a}\right)^3 \times \cos^3 l \times \left\{+ \sin \overline{|v-V|} + \sin \overline{|3\,(v-V)|}\right\} \\
&- 213{,}10 \times \left(\tfrac{A}{R}\right)^4 \left(\tfrac{r}{a}\right)^3 \times \cos\ l \times \left\{+ \sin \overline{|v-V|}\right\} \\
&- 0{,}83 \times \left(\tfrac{A}{R}\right)^5 \left(\tfrac{r}{a}\right)^4 \times \cos^4 l \times \left\{+ 2 \sin \overline{|2\,(v-V)|} + \sin \overline{|4\,(v-V)|}\right\} \\
&+ 1{,}42 \times \left(\tfrac{A}{R}\right)^5 \left(\tfrac{r}{a}\right)^4 \times \cos^2 l \times \left\{+ \sin \overline{|2\,(v-V)|}\right\}
\end{aligned}
$$

For $\frac{1}{a} \times$ Force Normal to Ecliptic,

$$
\begin{aligned}
&- 55952{,}34 \times \left(\tfrac{A}{R}\right)^3 \left(\tfrac{r}{a}\right) \times \sin l \times \left\{+ 1\right\} \\
&+ 426{,}20 \times \left(\tfrac{A}{R}\right)^4 \left(\tfrac{r}{a}\right)^2 \times \cos l\ \sin l \times \left\{+ \cos \overline{|v-V|}\right\} \\
&- 1{,}42 \times \left(\tfrac{A}{R}\right)^5 \left(\tfrac{r}{a}\right)^3 \times \cos^2 l\ \sin l \times \left\{1 + \cos \overline{|2\,(v-V)|}\right\} \\
&+ 0{,}57 \times \left(\tfrac{A}{R}\right)^5 \left(\tfrac{r}{a}\right)^3 \times \sin l \times \left\{+ 1\right\}
\end{aligned}
$$

By application of those formulæ, the following tables are formed

The commas here and in the columns of the tables occupy the same place as in the columns of Sections II and III

NUMERICAL LUNAR THEORY.

SECTION IV. PARTS 2 AND 3.

SOLAR GRAVITATIONAL FORCES.

PART 2 —NUMERICAL DEVELOPMENT OF SOLAR GRAVITATIONAL FORCES IN THE PLANE OF ECLIPTIC, FOR INSERTION IN EQUATIONS (10) AND (11),

FROM No 1 TO No 15, DETAILED RESULT OF EVERY STEP,

FOR THE REMAINING NOS, FINAL RESULTS ONLY

COLUMNS 32 TO 68

PART 3 —NUMERICAL DEVELOPMENT OF SOLAR GRAVITATIONAL FORCES NORMAL TO THE PLANE OF ECLIPTIC, FOR INSERTION IN EQUATION (12),

DETAILED RESULT OF EVERY STEP

COLUMNS 69 TO 72

TDRMS DDVDLOPDD AND CO DFTIOIENTS WITH DTI INLD VAI UI S OI UNIIS IO MUI LII I Y

Ref in Argum	A	3 $\left(\frac{1}{R}\right)$ ( ) U –	3 $\left(\frac{1}{R}\right)$ ( ) U	4 $\left(\frac{1}{R}\right)$ ( ) U	35 $\left(\frac{1}{R}\right)$ ( ) U –	36 $\left(\frac{1}{R}\right)$ ( ) U –	$\left(\frac{1}{R}\right)$ ( ) U	38 $\left(\frac{1}{R}\right)$ ( ) U	3) (( ) U	4 ((–) U t –
		O	O	O	O	C	O	(	S	C
3 4 5	D *l* D *l* *l*	544 – 938 68 49	554 8 88 75 8836 48 6 5	79 636 779 – 58 –	8 9 3/ 8 3	4 7 5444 95957 76936 19 6	6 95 8 6 488 5	63 – 9	9/57 136 1887 17 84	47 74(7 3 5)(58
6 9	D *l* D *S* D *l* – *S* *l* – *S* D	– 44 – 58 – 4 7 7	4464 397 5 5 336 54	5 – – 5 – 64 8	– 3 –	– 4395 7 66 – 64 7 – 679 7 8	11 5 4 4 9 54	3 9	13 9 8 6 7 4 6 69	– 33 – 539 739 – 4 8 – 75
   4 5	*l* *S* *f* – *l* 3 *l* 4 D – *l* *S*	6 6 3	9 4 3 444 5 9 5 8648	– 17 63 3 68	– 8 85	– 7 75 – 6 8 – 635 5 564	3 5 1 4 (775	(85	53 7 9 753 858 – 36783	4 465 – 56 – 4 47 – 61

TDRMS DDVDLOPDD &c— *l* *d*

Reference Argum	5 9 | (( )| U – –	5 C |+(( )| U – –	53 8l | V| U –	54 O 1 | V| U	55 9 | (V)| U t –	56 O | (I)| U l	57 9 |3(V)| U t	58 (1 |3(–1)| U –
	9	Ob	9	C	8	C 1	9I	C
3 4 5	4 5 84 43 4	– 945 6 7 47	– 7 4 3	– 3 7 3	34 3 968 985999 3539	– 9 356 84 (95698 988 44 35 75	8 89	8 89
6 7 8 9	9 4 3 –	4 6 3	4 65	4 99 35	8 4 4 36 3 8 35 8 7 55 6 55	8 4 8 36 35 35 49 (9 3 6 15	8 63	– 8 – 53
  4 5	– 4 – 39	3 3		4	4 6 3 398 74 4 3	78 6 5 397 74 4 583		

SECTION IV PART 2—SOLAR FORCES PARALLEL TO ECLIPTIC NUMERICAL INVESTIGATION FOR FIFTEEN ARGUMENTS

THE SINES OR COSINES OF THEIR RESPECTIVE ARGUMENTS IN THE SAME LINE

Reference for Argument	4	4	43	44	45	46	47	48	49	50.
	(ζ θ) Unit σ⁻⁷	(ζ − θ) Unit − ⁻⁷	(ζ θ) Unit = σ⁻⁷	(ζ θ)² Unit = σ⁻⁷	Sine ζ Unit = σ⁻⁷	Cosine ζ Unit = σ⁻⁷	Sine (ζ − θ) Unit = σ⁻⁷	Cosine (ζ θ) Unit = 8σ⁻⁷	Sine 3 (ζ) Unit σ⁻⁷	Cosine 3 (ζ −) Unit = ⁻⁷
	Sines.	Cosines.	Sines.	Cosines.	Sines.	Cosines.	Sine	Cosines.	Sines.	Cosines.
3 4 5	3 5 4543 548 333	924 50 a86 5 7 85	209 97 54 30	4 5 7	09538 + aa 58 4483 3706	99649 5 − 37 8 − 5989 488 9785	7665 438641 6393 7 799	986 4834 58 3 + 4568 8808	3a37959 646760 3333 3 59 4	9624985 33 00 − 58049 8 3 64384
6 7 8 9	642 3 882 558 50	− 335 08 303 − 880	48 43 26 5	6 4 7 4	9583 8 00 9858 7048 − 8 44	644 2888 2683 20044 58	0794 749 888 3542 39	24398 647 458 79738 49	352 9 a8953 a8 80 8a22 − 7886	59633 3248 3a244 77773 343
3 4 5	373 44 3 75 95 − 7373	888 48 58 97 3	− 79 8 43	5 3 4	− 5a55 − 9 228 907 366803	− 20 96 076 20 7 3 − 305	40 3775 77 8 4 07 7a387	8a339 4 79 7989 3 209	− 4884 − 3333 9387 8824 − 070608	− 79 9338 7682 548 678

Reference for Argument	59	6	6	6a.	62	64	65	66	67	68.
			CO-EFFICIENT MULTIPLYING COSINE OF ARGUMENT FOR EQUATION ( )				CO-EFFICIENT MULTIPLYING SINE OF ARGUMENT FOR EQUATION ( )			
	Sine 4 (− P) Unit σ⁻⁸	Cosine 4 (P) Unit σ⁻⁸	First Term. Unit = σ⁻⁷	Second Term Unit = σ⁻⁷	Third Term. Unit σ⁻⁷	Total for Equation ( ) Unit = σ⁻⁷	First Term Unit = σ⁻⁷	Second Term Unit = σ⁻⁷	Third Term. Unit = σ⁻⁷	Total for Equation ( ) Unit = σ⁻⁷
	Sines.	Cosines.	Cosines.	Cosines	Cosines.	Cosines.	Sines.	Sines.	Sines.	Sines
3 4 5			6849 95 566 3 4225 04 − 82447 − 248 9	06	− 8 − 6	6830 24 − 566 3 − 42 5 82448 − 248 9	a855 3 3746 87 8a842 206 5	06	5	855 3 3746 97 − 82843 206 5
6 7 8 9 0			4560 5 3 8 7 79 34 04 88 63	5a 50	−	4560 5 3 − 8 7 79 − 34 3 39 3	457 3 39 789 3 58 83 78 3	48 6		− 457 − 3 39 789 33 58 83 − 24 64
1 3 4 5	3	− 3	47 83 − 20 5 985 79 94		6 02	− 47 81 − 20 − 5 985 79 95	96 53 4 988 4 28	− 1		96 − 53 4 − 988 4 7

SECTION IV, PART 2—*continued* SOLAR FORCE, PARALLEL TO THE ECLIPTIC, INCLUDING RESULTS ONLY, FOR ALL ARGUMENTS

Reference for Argument	64 Coefficient of Cosine Eq (10) Unit = 10^{-7}	68 Coefficient of Sine, Eq (11) Unit = 10^{-7}
1	+ 26830,24	
2	− 5661,30	+ 2855 32
3	− 14225,20	+ 13746 97
4	+ 82448	− 82843
5	− 248,9	+ 206,5
6	+ 4560	− 4571
7	+ 5103	− 5139
8	− 817 79	+ 789,33
9	− 234,05	+ 156,83
10	+ 239,13	− 124,64
11	− 147,83	+ 96,21
12	− 1,20	− 12,53
13	− 15	+ 14
14	+ 985	− 988
15	+ 1279,95	+ 14,27
16	+ 65,11	− 70,35
17	− 962	+ 957
18	− 54	+ 53
19	+ 706,56	− 601,58
20	+ 249	− 249
21	+ 316	− 316
22	+ 357	− 368
23	− 0,63	+ 0,06
24	+ 323,52	− 329,37
25	− 15	+ 13
26		+ 7,20
27	− 6	+ 2
8	− 7	+ 6
29	− 19,18	+ 18,84
30	+ 216	− 217
31	+ 34,03	− 33,18
32	− 32,6	+ 31,6
33	− 1	+ 1
34	− 76	+ 75
35	+ 107	− 106
36	− 68	− 68
37	+ 43	+ 43
38	+ 14	+ 14
39	− 6	− 5
40	+ 229	− 229
41	+ 3	− 3
42		
43	− 0,78	+ 5,13
44	− 0,30	+ 0,40
45	+ 38,47	− 30,41
46	+ 30,01	+ 0 73
47	+ 19	− 19
48	− 16	+ 17
45	+ 47	− 47
50	− 4 08	+ 3,06

Reference for Argument	64 Coefficient of Cosine, Eq (10) Unit = 10^{-7}	68 Coefficient of Sine Eq (11) Unit = 10^{-7}
51	+ 78,6	+ 28,1
5–		
53		
54	− 14	+ 13
55	− 33 99	+ 10,85
56	+ 1,17	− 1 05
57	+ 19,71	− 20,06
58	+ 1	− 1
59	+ 0,89	+ 0,39
60	− 1	− 1
61		
62		
63	− 1	+ 1
64	+ 15	− 15
65	+ 6,38	− 6 14
66	− 3 07	+ 2,95
67	+ 15	− 16
68		
69	− 1,07	+ 1,07
70	+ 0,5	− 0,2
71	+ 0,11	+ 0,53
72	+ 19	− 19
73	+ 1	
74	+ 2 21	− 2,15
75	+ 1,60	+ 12,84
76	+ 2	+ 1
77	+ 2	+ 1
78	+ 0,07	− 0,45
79	− 0,04	− 0,59
80		
81	− 2	+ 2
82	+ 1,11	− 1,09
83	− 6	+ 6
84	+ 3,44	+ 0,33
85	− 7	+ 7
86	+ 0,06	+ 0,08
87	+ 21	− 21
88	+ 0,23	+ 0,15
89	+ 0,09	− 0 05
90		
91		
92		+ 0,6
93	+ 2	+ 4
94	− 3,76	+ 3,73
95	− 0,50	+ 0 49
96	− 2	+ 2
97	+ 0,03	+ 0,02
98	− 0,20	+ 0,39
99	+ 0 01	− 0,01
100	− 0,05	+ 0,04

Reference for Argument	64 Coefficient of Cosine, Γq (10) Unit = 10^{-7}	68 Coefficient of Sine Eq (11) Unit = 10^{-7}
101	− 1 64	− 0,06
102	− 0,27	+ 0,24
103	+ 1,52	− 1,07
104	+ 4	+ 12
105	+ 1	− 1
106		− 0,5
107	− 0,08	+ 0,16
108	+ 0,13	− 0,07
109	+ 1	− 2
110		
111	− 1	+ 1
112	+ 7	− 8
113		
114		
115		
116	− 0,12	+ 0,09
117		
118		
119	− 0,02	+ 0,03
120		
121	+ 12	− 12
122		
123	+ 11	− 11
124	− 0,51	− 0,58
125		
126	− 0,1	
127	+ 7	− 7
128		
129	+ 2	− 2
130		+ 1
131	+ 4	− 5
132	+ 4	− 5
133		
134	− 1	+ 2
135	+ 3	− 2
136		
137	+ 1	− 1
138	+ 1	− 2
139	+ 1	− 2
140	+ 1	− 1
141	− 1	− 1
142	− 6	− 5
143		
144		
145	− 0,08	+ 0,01
146		
147		
148	− 0 22	+ 0,18
149		
150		

Reference for Argument	64 Coefficient of Cosine Iq (10) Unit = 10^{-7}	68 Coefficient of Sine, Γq (11) Unit = 10^{-7}
164	+ 0,7	
165		+ 1
167		− 1
173	− 0,2	
174	+ 2	+ 2
179	− 0,21	+ 0,19
182	+ 0,6	
184		− 1
188	− 0,6	+ 2
189		− 1
194	+ 4	− 4
195	+ 1	
196	+ 8	+ 8
197	− 0,8	
198	+ 0,78	+ 0,05
200		0, 2
201	+ 0,13	+ 0,13
202		− 1
204	+ 1	− 1
207	+ 1	
209	+ 2	+ 2
212	− 0,03	− 0,02
217	+ 1	− 2
219	+ 0,08	− 0,07
220	− 1	
221	+ 1	− 1
224		+ 1
227	− 0,31	+ 0,31
230		+ 1
231		+ 1
234	+ 1	
244	− 1	
251		− 1
253	+ 1	− 1
256		+ 1

SECTION IV PART 3 —SOLAR FORCE NORMAL TO THE ECLIPTIC FOR ALL ARGUMENTS

Reference for Argument	69 Argument	70 First Term Unit = 10^{-7}	71 Second and (Third) Terms Unit = 10^{-7}	72 Total for Equation (12) Unit = 10^{-7}
301	f	− 5013,20	(− 0,07)	− 5013,27
302	$f + l$	− 138		− 138
303	$f - l$	+ 408		+ 408
304	$2D - f$	− 185,9	(− 0,1)	− 185,8
305	$2D + f - l$	− 30		− 30
306	$2D - f - l$	− 66		− 66
307	$2D + f$	− 10		− 10
308	$f + 2l$	− 6		− 6
309	$2D - f + l$	− 5		− 5
310	$f - 2l$	+ 5,01		+ 5,1
311	$2D - f - S$	− 13,9		− 13,9
312	$2D - f - 2l$	− 2,6		− 2,6
313	$2D + f + l$	− 1		− 1
314	$2D - f + S$	− 1,2		− 1,2
315	$2D + f - l - S$	− 2		− 2
316	$2D + f - S$	− 1		− 1
317	$2D - f - l - S$	− 5		− 5
318	$4D - f - l$	− 1		− 1
319	$f + l + S$	− 4		− 4
320	$f + S$	− 124,7		− 124,7
322	$f - l + S$	+ 12		+ 12
323	$D + f$	+ 1	− 19	− 18
324	$f + l + S$	− 3		− 3
325	$f - l - S$	+ 8		+ 8
326	f	− 127,8		− 127,8
327	$D - f$	+ 2	+ 18	+ 20

Reference for Argument	69 Argument	70 First Term Unit = 10^{-7}	71 Second and (Third) Terms Unit = 10^{-7}	72 Total for Equation (12) Unit = 10^{-7}
331	$3f - l$	+ 1		+ 1
333	$2D - 3f$	− 0,7		− 0,7
335	$2D + f - l + S$	− 1		− 1
337	$2D + f - 2l$	− 1,0		− 1,0
342	$2D - f - l + S$	− 1		− 1
343	$2D - f - 2S$	− 1		− 1
352	$D + f + l$		− 1	− 1
354	$D - f + l$	+ 0,1	− 1,0	− 0,9
359	$D - f - l$	+ 0,1	− 1,6	− 1,5
361	$3D - f$		− 1	− 1
363	$f - 2l - S$	+ 0,1		+ 0,1
365	$f - 2l + S$	+ 0,2		+ 0,2
380	$D + f - l$		+ 3,3	+ 3,3
384	$2D - f + 2S$	+ 0,01		+ 0,01
391	$f - 2S$	− 3		− 3
392	$f + 2S$	− 3		− 3
414	$2D - f - 2l + S$	− 0,1		− 0,1
431	$D + f - l + S$	+ 0,1	+ 0,08	+ 0,09
433	$D - f + l + S$	+ 0,01	− 0,01	
435	$D - f - l + S$		+ 0,04	− 0,04
454	$D - f - S$	+ 1		+ 1
480	$D + f - S$	− 1		− 1

It may be instructive here to examine the relation which must exist between—the movement of a small body in a slightly disturbed circular orbit round a center of attraction—and the forces, radial and tangential, which act on that small body We will put E for the mass of the central body or its attractive accelerating-force at distance 1, $\frac{E}{r}$ its force at distance r, nt the mean angular motion in the time t, $r = a\{1 + 2A \cos \overline{pt}\}$, v the angle described round the center $= \overline{nt} + 2B \sin \overline{pt}$, where p may be any number The multiple 2 is introduced only to avoid many fractions. The investigations are carried only to the first powers of A and B

First Investigation of the forces necessarily corresponding with the geometrical assumptions,

$$\cos v = \cos \overline{nt} - 2B \sin \overline{nt} \sin \overline{pt}, \quad \sin v = \sin \overline{nt} + 2B \cos \overline{nt} \sin \overline{pt}$$

$$x = r \cos v = a\{\cos \overline{nt} + (A + B) \cos \overline{(n+p)t} + (A - B) \cos \overline{(n-p)t}\},$$

$$y = r \sin v = a\{\sin \overline{nt} + (A + B) \sin \overline{(n+p)t} + (A - B) \sin \overline{(n-p)t}\}$$

The forces required to maintain these ordinates are,

$$\text{Force in } x = \frac{d^2x}{dt^2} = a\{-n^2 \cos \overline{nt} - (n+p)^2 (A + B) \cos \overline{(n+p)t}$$
$$- (n-p)^2 (A - B) \cos \overline{(n-p)t}\}$$

$$\text{Force in } y = \frac{d^2y}{dt^2} = a\{-n^2 \sin \overline{nt} - (n+p)^2 (A + B) \sin \overline{(n+p)t}$$
$$- (n-p)^2 (A - B) \sin \overline{(n-p)t}\}$$

$$\text{Force in } r = \frac{x}{r} \text{ force in } x + \frac{y}{r} \times \text{force in } y =$$

$$a\{-n^2 \cos^2 \overline{nt} - (n+p)^2 (A + B) \cos \overline{nt} \cos \overline{(n+p)t}$$
$$- (n-p)^2 (A - B) \cos \overline{nt} \cos \overline{(n-p)t} + 2n^2 B \sin \overline{nt} \cos \overline{nt} \sin \overline{pt}\},$$
$$+ a\{-n^2 \sin^2 \overline{nt} - (n+p)^2 (A + B) \sin \overline{nt} \sin \overline{(n+p)t}$$
$$- (n-p)^2 (A - B) \sin \overline{nt} \sin \overline{(n-p)t} - 2n^2 B \sin \overline{nt} \cos \overline{nt} \sin \overline{pt}\},$$
$$= + a\{-n^2 - (n+p)^2 (A + B) \cos \overline{pt} - (n-p)^2 (A - B) \cos \overline{pt}$$
$$= a\{-n^2 - [(2n^2 + 2p^2) A + 4np B] \cos \overline{pt}\}$$

From this must be subtracted the ordinary gravitational force as estimated from the central body, corresponding to the position of the disturbed body at that moment, or $-\frac{E}{r^2}$, or $-\frac{E}{}(1 - 4A \cos \overline{pt})$ As the object of our investigation is, to discover the relations of the small terms as distinguished from the large terms, we must suppose the large terms independently to satisfy the equations, or $\{a \times + n^2\} = \frac{E}{a^2}$, or $E = + a^3 n^2$, and the quantity to be applied is $a\{+ n^2 - 4n^2 A \cos \overline{pt}\}$

This leaves for the real radial perturbational force,

$$a \times \{-(6n^2 + 2p^2) A - 4np B\} \times \cos \overline{pt}$$

We shall hereafter call this quantity $n^2 a \text{ R} \cos \overline{pt}$

Tangential Force $= \frac{x}{r} \times$ force in $y - \frac{y}{r} \times$ force in $a =$

$$a \{ - n^2 \sin\ nt\ \cos\ nt - (n+p)^2\ (\mathrm{A} - \mathrm{B})\ \cos\ \overline{nt}\ \sin\ \overline{(n+p)\,t}$$
$$- (n-p)^2 (\mathrm{A} - \mathrm{B})\ \cos\ \overline{nt}\ \sin\ \overline{(n-p)\,t} + 2n^2\,\mathrm{B} \sin\ {}^2nt \sin\ pt$$
$$+ a \{ + n^2 \sin\ \overline{nt}\ \cos\ \overline{nt} + (n+p)^2\ (\mathrm{A} + \mathrm{B})\ \sin\ \overline{nt}\ \cos\ \overline{(n+p)\,t}$$
$$+ (n-p)^2 (\mathrm{A} - \mathrm{B})\ \sin\ \overline{nt}\ \cos\ \overline{(n-p)}\,t + 2n^2\,\mathrm{B} \cos\ {}^2\overline{nt} \sin\ \overline{pt}$$
$$= + a \{ - (n+p)^2\ (\mathrm{A} + \mathrm{B})\ \sin\ \overline{pt} + (n-p)^2 (\mathrm{A} - \mathrm{B})\ \sin\ \overline{pt} + 2n^2\,\mathrm{B}\ \sin\ \overline{pt} \}$$

We shall call this quantity $n^2\ a\mathrm{T}\ \sin\ \overline{pt}$

Second We have therefore the two equations—

$$- (6n^2 + 2p^2)\ \mathrm{A} - 4\,np\ \mathrm{B} = n^2\,\mathrm{R}$$
$$- 4\,np\ \mathrm{A} - 2p^2\ \mathrm{B} = n^2\,\mathrm{T}$$

And from these we find—

$$\mathrm{A} = \frac{n^2}{n - p^2} \left\{ \frac{1}{2}\,\mathrm{R} - \frac{n}{p}\,\mathrm{T} \right\} \qquad \mathrm{B} = \frac{n}{n - p^2} \left\{ - \frac{n}{p}\,\mathrm{R} + \frac{3n^2 + p}{2p}\,\mathrm{T} \right\}$$

These expressions exhibit the numerical values of the perturbations of elliptical elements A and B which will be produced by disturbing forces R and T It is particularly to be remarked, that both disturbances are very large when p is nearly equal to n, that is, when the periodic time of either or both disturbances is nearly the same as the periodic elliptic time, and also, that the disturbances are very large when p is very small, that is, when the periodic time of the disturbances is very long

These remarks may be borne in mind, in estimating the effects of such forces as those in page 66

NUMERICAL LUNAR THEORY

SECTION V

VERIFICATION OF SECTIONS II, III, AND IV

AND

MODIFICATION OF THE ASSUMED VALUE OF $\frac{a}{r}$

NUMERICAL LUNAR THEORY

SECTION V—VERIFICATION OF CALCULATIONS IN SECTIONS II, III, AND IV, AND MODIFICATION OF THE ASSUMED VALUE OF $\frac{a}{r}$

I have carefully verified the complicated calculations of Sections II and III, by cross multiplication of the printed numbers, in directions differing generally from those which have been used in preparing the numbers, and (in some instances) by recalculation of numbers Of the cross multiplications, twelve have applied to principal terms, and three to differential co efficients My examination included all results as far as Argument No 15 And I have found excellent agreement between the old and the new results, and I have the most perfect confidence in the accuracy of all the Columns 1–23 and 24–29

It is to be remarked that the examination now described does not apply to Section IV But that work has been closely examined by an able superintendent, and I have inspected various portions of it, and I confide fully in its general accuracy

The calculations of Section V have been repeated in the original form, with slight variation of elements, and I have no doubt of their accuracy

I now advert to a trifling change of numbers, the necessity for which has been suggested by the examination to which I have alluded

It will be remarked in Equation (10) of Section I, that every term of the orbital expression contains, as factor, the quantity $\left(\frac{r}{a}\right)^2$, and the one term which represents the Terrestrio Lunar Gravitational Force contains the factor $\frac{a}{r}$ For the annihilation of terms which is required for producing perfect theoretical solution of the Equation, it is necessary that, when both expressions have been expanded in multiples of "constant," "cos $\overline{|1|}$," "cos $\overline{|2D-1|}$," &c, the co-efficients attached to each separate argument in the two expressions should destroy each other Now, in a few instances, this destruction is not complete, and the imperfection is to be assigned to an algebraical circumstance

When the series contained in Column 1 had been adopted as expressing $\frac{a}{r}$ (the quantity which refers immediately to Delaunay's theory, and which enters into the gravitational terms, and from which $\left(\frac{a}{r}\right)^2$ is easily found) it was matter of some difficulty to derive from it the series for $\frac{r}{a}$ and that for $\left(\frac{r}{a}\right)^2$ (the quantity which enters into the orbital terms) The method selected was to express $\frac{a}{r}$ by $1 + \left(\frac{a}{r} - 1\right)$ $\left\{\text{where } \left(\frac{a}{r} - 1\right) \text{ is a small quantity of the order of excentricity}\right\}$, and $\left(\frac{a}{r}\right)^2$ by the finite series $1 + 2\left(\frac{a}{r} - 1\right) + \left(\frac{a}{r} - 1\right)^2$, and to express $\frac{r}{a}$ by the infinite series $1 - \left(\frac{a}{r} - 1\right) + \left(\frac{a}{r} - 1\right)^2 - \&c$, and $\left(\frac{r}{a}\right)^2$ by $1 - 2\left(\frac{a}{r} - 1\right) + 3\left(\frac{a}{r} - 1\right)^2 - 4\left(\frac{a}{r} - 1\right)^3 + \&c$

This series was carried to the 5th power of $\left(\frac{a}{r} - 1\right)$, and with this value for $\left(\frac{r}{a}\right)^2$, the numbers in Section II are formed The validity of these expansions was tested by forming the quantities—

$$\frac{a}{r} \times \frac{r}{a}, \text{ which ought to equal } 1$$

$$\left(\frac{a}{r}\right)^2 \times \left(\frac{r}{a}\right)^2, \text{ which ought to equal } 1$$

The former of these conditions is sensibly satisfied, but the latter is not Omitting some very trifling discordances (such as will occur in computing decimal quantities by two different methods) the product of $\left(\frac{a}{r}\right)^2$ by $\left(\frac{r}{a}\right)^2$ gives—

$$0\,9999992 + 0000036 \cos\overline{|l|} + 0000034 \cos\overline{|2D - l - S|}$$
$$+ 0000030 \cos\overline{|2f - l|} + 0000025 \cos\overline{|4D - l|}$$

It appears certain that these terms have arisen from insufficient extension of the powers of $\left(\frac{a}{r} - 1\right)$

Equality may be produced by small changes, in the expression for $\frac{a}{r}$, or in that for $\left(\frac{r}{a}\right)^2$, or in both

Now it is to be remarked that the series for $\frac{a}{r}$ is not assumed as a certain and unalterable series It is expressly assumed as liable to alteration, and the entire primary object of this Theory is to discover what alteration ought to be made

We are then at liberty to adopt either of the three systems of change just mentioned An inspection of the computed columns will show that it is far most convenient that the value of $\left(\frac{r}{a}\right)^2$ thus far used should be retained, and that the whole change should be thrown on the expression for $\left(\frac{a}{r}\right)$ This will be done by applying, to the last figures of the assumed numbers in Column 1 the following corrections —

$$+ 4 - 18 \cos\overline{|l|} - 17 \cos\overline{|2D - l - S|} - 15 \cos\overline{|2f - l|} - 12 \cos\overline{|4D - l|},$$

and thus, in subsequent calculations, the following numbers are to be used instead of those for the same Nos in Column 1 —

$$\text{Reference No } 1 \begin{Bmatrix} + \;0\,0000004 \text{ for } \left(\frac{a}{r} - 1\right) \\ + \;1\,0000004 \text{ for } \frac{a}{r} \end{Bmatrix}$$

No 2	+	0545077
No 8	−	4226
No 12	−	2084
No 14	+	1833

The numbers, thus modified, will be used when necessary in the following calculations And when, in the course of the Theory, corrections distinguished by the symbols δg_2, δg_3, &c shall be found, these corrections are to be applied to the numbers modified as is shown above

On inserting the corrections just found in the Developments of $-M \frac{a}{r} \cdot \cos 2l$ and $-M \left(\frac{a}{r}\right)^2 \sin l$, Section III, it is found that the only addition required to $-M \cdot \frac{a}{r} \cos 2l$, expressed in units of 10^{-7}, is—

$$-4 + 18 \cos l + 17 \cos \overline{|2D - l - S|} + 15 \cos \overline{|2f - l|} + 12 \cos \overline{|4D - l|},$$

and that no term in the addition required to $M \left(\frac{a}{r}\right)^2 \sin l$ amounts to 0000001

Specimens of two of the skeleton forms employed in these calculations are attached

NUMERICAL LUNAR THEORY, FORM (II)

Recalculation of Terms forming Equations (10) *and* (11)

Formation of						*Multiplication of*	*by*	*and*
Reference No	Factor ½ or 2	Sine or Cosine	Trigon Terms: Special / General / Sum / Difference	Co efficient: Special / General / Product	Logarithm: Special / General / Sum			
				5				

NUMERICAL LUNAR THEORY, FORM (IV)

Recalculation of Terms for forming Equations (10) *and* (11) *as far as Reference No* 15

H	Formation of $\left(\frac{r}{a}\cos 1\right)^2$, Column 14, and of $\frac{1}{2}\frac{d^2}{dt^2}\left\{\left(\frac{r}{a}\cos 1\right)^2\right\}$ Column 22 by the Product Column 6 × Column 12, and further Product × ½ Movement of Argument														
Nos of Columns	Reference No m Cols	Sines or Cosines	Factors ½ 1 or 2	Trig Terms: Special / General; Sum / Difference; Adopted Argument	Co efficients: Special / General; Product {Movement of Adopted Argument} M o A A repeated o 5; Second Product	Logarithms: Sum Log M o A A Log repeated +9 6989700; Second Sum									Number of Decimals in Logarithms
6 12		C C					C C		+0 5	+9 6989700	C C		+0 5	+9 6989700	
6 12		C C			+0 5	+9 6989700	C C		+0 5	+9 6989700	C C		+0 5	+9 6989700	
6 12		C C			−0 5	−9 6989700	C C		−0 5	−9 6989700	C C		+0 5	−9 6989700	

NUMERICAL LUNAR THEORY.

SECTION VI

INVESTIGATION AND APPLICATION OF NUMERICAL VALUE OF M,

AND

EXHIBITION OF REMAINING DISCORDANCES BETWEEN ORBITAL AND GRAVITATIONAL FORCES,

OR,

UNCORRECTED NUMERICAL ERRORS OF THE THREE FUNDAMENTAL EQUATIONS.

Columns 73 to 77

NUMERICAL LUNAR THEORY

Section VI.—Investigation and Application of the value of M and Exhibition of the remaining Discordances, or, uncorrected Numerical Values of the three Fundamental Equations

In every case of orbital motion under the action of centripetal force, there must be an equation between the mean periodic time, the mean radius vector or the mean parallax, and the masses of the attracting bodies. We have assumed the first and second of these, and the third consists in the use of Equation 10, combined with or corroborated by Equation 12, and we have therefore the means of determining the value of $\frac{e+\mu}{a^3}$ (which we have called M), or the masses of the attracting bodies.

From our assumed (Delaunay's) values of the co efficients of the terms in the series representing $\frac{a}{r}$, v, and sine l (Columns 1, 15, and 24) we have (Section II, Part 2) formed Column 23, to be compared with M × Column 30 + Column 64, and (Section II Part 3) Column 29, to be compared with M × Column 31 + Column 72, and not only ought these equations to hold generally, but they ought to hold specially for every term with different arguments, and if we employ each and every one to determine the value of M, the separate resulting values for M ought to be practically in accord. The following statement will show the result of this comparison for some of the terms. (It is to be remarked that the numbers in Columns 64 and 72 are considered to be theoretically correct, and numerically accurate.)

For Argument 1, − 9960060 = M × − 9959740 + 26830.
For Argument 2, − 549570 = M × − 542808 − 5661
For Argument 301, − 902822 = M × − 895367 − 5013

The values obtained for M are respectively 1 002726, 1 002021, 1 002721

Many other terms in the series of $\frac{a}{r}$ have been tried, and have produced results still more irregular. These irregularities are, in fact, the result of apparent errors in the assumed values of $\frac{a}{r}$ and l, which will be seen distinctly in the last Table of this Section, Columns 74 and 77.

I may remark that the discordance for Argument 2 would be removed by multiplying the original co efficient in Column 1 by 999770. But the same factor will not produce accordance when applied to the co efficients of other terms.

The discordance of these numbers has given me much anxiety. It is evident that there is serious error, in the terms mainly dependent on $\frac{a}{r}$, either in M. Delaunay's calculations or in my own. My computations have been made independently three times, under circumstances which I thought would insure their accuracy.

Viewing the magnitude of the term on which the deductions for Argument 1 are founded, and the general simplicity of the investigations for Arguments 1 and 301 as compared with those for Argument 2, I have determined to base my value of M entirely on Arguments 1 and 301.

At the same time, I think it desirable to leave opening for further change I shall therefore use

$$M = 1\ 0027250 + \delta M$$

The tables which follow exhibit the result of application of this element

No change is made in the Columns 23 and 29

In the first column of each set of parallel columns in the following tables, applying to ecliptic radial forces, Column 73 of terrestro-lunar gravitational force is formed by multiplying the numbers of Column 30 as far as No 50 by 1 0027250 (the correction from Section V having been introduced) reserving the multiple of δM for a following column, the solar gravitational force is taken from Column 64, and the orbital force from Column 23 Column 74 is formed by adding with proper signs Columns 73, 64, and 23, and here the multiplier of δM is introduced

In the second column of each set of parallel columns, applying to disturbance of ecliptic areas, the terrestro lunar force does not enter, and M therefore will not appear For the gravitational solar transversal force we refer to Column 68, and for the orbital force to Column 18, their sum is shown in Column 75

For the third set, Column 76 is formed by multiplying Column 31 by 1 0027250 (in the terms following No 350, it is sufficient to use the numbers of Column 31, without multiplication, by 1 0027250), the solar gravitational force is taken from Column 72, and the orbital force from Column 29, and Column 77 is formed by adding with proper signs Columns 76, 72, and 29, and here δM is introduced

The residual quantities in Columns 74, 75, 77, are the Uncorrected Values of the Three Fundamental Equations, which we must endeavour to correct by changes in the assumed numerical values of arguments and co efficients in Columns 1, 15, and 24

NUMERICAL LUNAR THEORY

UNCORRECTED NUMERICAL VALUES

Reference to Argument	Forces in Ecliptic Radius: 73 = − M × Col 30 (modified by Section V)	Forces in Ecliptic Radius: 74 = 23 − 64 − 73 = Numerical Term of Equation (10)	Transversal Forces: 75 = 18 − 68 = Numerical Term of Eq (11)
1	− 9986884,3	− 5,6 + δM × 9960	
2	− 544269	+ 360 + δM × 0543	+ 192 68
3	− 99428,7	− 1563,4 + δM × 0099	− 725,9
4	− 84477	− 911 + δM × 0085	− 52,
5	− 29852 8	+ 607,3 + δM × 0030	+ 345,2
6	− 9259	− 647 + δM × 0009	− 385
7	− 5574	− 726 + δM × 0006	− 429
8	− 4208	+ 71 + δM × 0004	+ 39 1
9	− 3066,7	− 1080,2 + δM × 0003	− 506 4
10	+ 2733,0	+ 627 4 − δM × 0003	+ 298 5
11	+ 2661 7	+ 450,8 − δM × 0003	+ 234,9
12	+ 5361	− 17 − δM × 0005	− 0 11
13	− 1857	+ 275 + δM × 000.	+ 141
14	− 1506	− 2774 + δM × 0002	− 1494
15	+ 1.43,3	− 235 8 − δM × 0001	− 9,41
16	+ 1101 9	− 1633,9 − δM × 0001	− 783,5
17	+ 1001	− 407 − δM × 0001	− 229
18	− 903	− 924 + M × 0001	− 551
19	+ 832 3	+ 237,7 − δM × 0001	− 26 35
20	− 827	− 550 + δM × 0001	− 269
21	− 591	− 940 + δM × 0001	− 514
22	− 617	− 2833 + δM × 0001	− 1334
23	− 438,4	− 16,7	− 8
24	+ 2861,2	− 33 4 − δM × 0003	+ 3 20
25	− 302	− 352	− 196
26	+ 297,8	+ 200 8	− 103,3
27	+ 286	+ 231	0,0
28	+ 269	+ 201	+ 117
29	+ 963,2	+ 17 2 − δM − 0001	− 10,3
30	− 224	− 247	− 138
31	+ 218,4	− 23,5	− 9,6
32	− 132,0	− 28,9	− 11 5
33	− 121	+ 31	+ 7
34	+ 126	+ 318	+ 170
35	− 91	− 1001	− 502
36	+ 59	+ 281	+ 151
37	− 63	− 1459	− 501
38	− 54	− 434	− 155
39	− 46	− 243	− 148
40	− 43	+ 351	+ 149
41	+ 98	+ 4	+ 3
42	− 39	− 66	− 34
43	− 38,1	− 54 3	− 24,5
44	+ 37,3	+ 65,8	+ 33,28
45	+ 37,1	+ 61,0	− 8,39
46	+ 35,9	− 35 5	− 3,12
47	− 35	− 425	− 173
48	− 1001	+ 3 + M × 0001	+ 3
49	− 38	− 909	− 341
50	+ 29,6	+ 5,1	+ 3,3
51	− 39783,3	+ 61 5 + δM × 0040	+ 35,6

OF EQUATIONS (10) AND (11)

Reference to Argument	Radial Forces 74 = 23 − 64 − 30 = Numerical Term of Eq (10)	Transversal Forces 75 = 18 − 68 = Numerical Term of Eq (11)	Reference to Argument	Radial Forces 74 = 23 − 64 − 30 = Numerical Term of Eq (10)	Transversal Forces 75 = 18 − 68 = Numerical Term of Eq (11)	Reference to Argument	Radial Forces 74 = 23 − 64 − 30 = Numerical Term of Eq (10)	Transversal Forces 75 = 18 − 68 = Numerical Term of Eq (11)	Reference to Argument	Radial Forces 74 = 23 − 64 − 30 = Numerical Term of Eq (10)	Transversal Forces 75 = 18 − 68 = Numerical Term of Eq (11)
51	+ 61,5		101	− 3,68	− 0,0	151	− 18	− 11	201	+ 2	
52	+ 24	− 17	102	− 0,55	− 0,05	152	− 86	− 9	202	+ 0,12	− 0,15
53	+ 73	− 40	103	+ 5,0	− 0,9	153	+ 31	+ 16	203	+ 8	+ 4
54	+ 171	+ 126	104	− 405	− 143	154	+ 3		204	+ 2,1	+ 0,7
55	− 38,6	+ 2,82	105	− 133	− 57	155	+ 7		205	− 5	− 3
56	+ 1,7	− 0,5	106	− 0,4	+ 1,0	156	− 13	− 5	206	+ 2	+ 1
57	+ 8 8	+ 1,46	107	− 1,6	− 0,2	157	− 43	− 36	207	+ 3,1	− 2
58	+ 25	+ 31	108	+ 0,11	+ 0,02	158	− 2	− 3	208	+ 2	
59	− 76,1	+ 38,3	109	− 100	− 46	159	− 7	− 11	209	− 2	− 4
60	− 131	− 65	110	− 42	− 19	160	− 1		210	− 1	+ 6
61	+ 68	+ 27	111	+ 94	+ 38	161	− 5	− 9	211	+ 2	+ 3
62	− 15	− 7	112	− 277	− 75	162	− 36	− 12	212		
63	+ 92	+ 43	113	+ 37	− 16	163	+ 2	− 1[illegible]	213	+ 0,03	+ 0,02
64	− 153	− 71	114	+ 9	+ 3	164	+ 1,6	− 0,9	214	+ 4	− 8
65	+ 25,6	+ 12,0	115	+ 61	− 38	165	− 11	− 10	215		− 2
66	+ 0,4	− 0,6	116	+ 2,8	+ 0,1	166	− 75	− 51	216	− 1,2	+ 1
67	+ 9	+ 8	117	+ 27	+ 16	167	+ 30	+ 15	217	+ 2	+ 16
68	− 79	− 40	118	− 20	+ 8	168	− 6	− 3	218	− 2	− 4
69	+ 2,3	− 0,6	119	+ 0,2	+ 0,6	169	− 22	− 16	219	− 1,2	− 0,8
70	+ 7,1	+ 3,7	120	− 14	+ 2	170	+ 6	− 2	220	− 2	− 6
71	+ 22,3	− 11,8	121	− 358	− 109	171	+ 14	+ 12	221	− 2	+ 12
72	+ 55	+ 15	122	− 75	− 33	172	+ 17	+ 12	222	− 1	− 3
73	− 12	− 9	123	− 368	− 71	173	− 2,5	− 0,3	223	+ 1	− 2
74	+ 1,9	+ 1,2	124	− 1,3	+ 0 30	174	− 9	− 3	224	− 2	− 8
75	+ 16,0	− 1,7	125	− 2,3	+ 0,5	175	− 17	− 15	225		+ 2
76	+ 13	+ 9	126	− 1,4	− 0,3	176	+ 20	− 6	226	+ 2	+ 8
77	+ 35	+ 11	127	− 89	− 37	177	− 8	− 8	227	+ 0,31	− 0,31
78	− 6,0	− 3,2	128	− 15	− 6	178	+ 1	− 1	228	− 2	− 7
79	+ 9,4	+ 5,2	129	− 28	+ 9	179	− 3	+ 0,48	229	+ 1	+ 4
80	+ 16	+ 9	130	+ 12	+ 4	180	− 14	− 12	230	+ 8	− 28
81	− 16	− 11	131	− 155	− 1	181	+ 2	− 1,0	231		− 6
82	− 4,7	− 2,2	132	− 91	− 16	182	− 28	− 19	232	− 1	− 3
83	+ 159	+ 57	133	+ 2		183	+ 12	+ 5	233		+ 5
84	+ 23,3	− 2,4	134	+ 63	+ 27	184	+ 29	+ 26	234	− 7	− 21
85	+ 193	+ 80	135	− 52	+ 6	185		− 0,8	235		+ 3
86	+ 9,8	− 5,24	136	+ 46	+ 24	186	− 5	− 2	236	+ 1	+ 3
87	− 1	− 9	137	− 41	− 14	187	+ 9	+ 5	237	+ 2	+ 12
88	+ 23,1	− 11,9	138	− 23	+ 29	188	+ 1		238	+ 1	+ 2
89	+ 4,5	− 2,2	139	− 33	+ 8	189	− 6	− 2	239	+ 1	+ 4
90	− 23	− 13	140	− 47	+ 14	190	− 14	− 7	240	− 1	+ 6
91	− 1		141	+ 15	+ 5	191	− 3		241	− 1	+ 7
92	+ 20,8	+ 10,8	142	+ 38	+ 22	192	+ 8	+ 18	242	− 2	− 6
93	− 2	− 1	143	+ 18	+ 9	193	+ 9	+ 24	243		− 7
94	− 2,0	+ 0,16	144	− 4	+ 7	194	− 17	− 14	244	+ 1	+ 4
95	+ 0,4	− 0,3	145	+ 2,0	+ 0,31	195	− 18	− 19	245	− 4	− 11
96	+ 38	+ 16	146	− 16	− 10	196	− 48	− 25	246	− 1	
97	− 2,3	+ 1,0	147	− 2	+ 0,15	197	− 11,0	+ 3,2	247	− 1	
98	+ 3,2	+ 1,5	148	+ 0,9	− 0,2	198	+ 2,7	+ 0,50	248	− 1	
99	− 2,21	+ 1,09	149		+ 20	199	+ 4,8	+ 1,4	249	− 1	
100	− 0,7	− 0,3	150	− 69	− 24	200	− 1,5	− 1,2	250	− 1	

UNCORRECTED NUMERICAL VALUES OF EQUATION (12)

Reference to Argument	Forces Normal to Ecliptic: 76 = − M × Col 31	Forces Normal to Ecliptic: 77 = 29 − 72 − 76 = Numerical Term of Equation (12)
301	− 897806 74	− 2,47 + δM × 0899
302	− 98060	+ 11 + δM × 0098
303	− 402	+ 5
304	− 23603	+ 18 + δM × 0024
305	− 18734	− 69 + δM × 0019
306	− 226	+ 45
307	− 14384	− 60 + δM × 0014
308	− 9031	+ 25 + δM × 0009
309	− 2699	+ 12 + δM × 0003
310	+ 840,3	+ 11,8 − δM × 0001
311	− 982	+ 13 + δM × 0001
312	− 591,8	− 9,8 + δM × 0001
313	− 2553	− 64 + δM × 0003
314	+ 523,1	+ 4,0 − δM × 0001
315	− 825	− 4 + δM × 0001
316	− 946	− 50 + δM × 0001
317	− 19	− 1
318	− 538	+ 71 + δM × 0001
319	− 596	− 38 + δM × 0001
320	+ 429,3	− 14,7
321	+ 41	+ 4
322	+ 5	− 14
323	+ 507	+ 21 − δM × 0001
324	+ 522	+ 17 − δM × 0001
325	− 16	+ 7
326	− 126,7	− 7,4
327	− 11	− 7
328	− 781	+ 21 + δM × 0001
329	− 414	+ 87
330	− 461	− 104
331	+ 368	− 18
332	− 315	− 26
333	− 166,6	+ 1,4
334	− 271	− 1
335	+ 191	− 55
336	− 135	+ 17
337	− 139,3	+ 11,8
338	− 78	+ 187
339	− 327	− 24
340	− 121	− 10
341	+ 179	+ 10
342	− 19	+ 20
343	− 34	+ 5
344	− 176	− 23
345	− 211	− 77
346	+ 4	− 4
347	+ 73	+ 18
348		
349	− 78	+ 2
350	− 103	− 7

Reference to Argument	Forces Normal to Ecliptic: 77 = 29 − 72 − 76 = Numerical Term of Eq (12)	Reference to Argument	Forces Normal to Ecliptic: 77 = 29 − 72 − 76 = Numerical Term of Eq (12)	Reference to Argument	Forces Normal to Ecliptic: 77 = 29 − 72 − 76 = Numerical Term of Eq (12)
351	− 5	401	− 10	451	− 1
352	+ 13	402	− 8	452	+ 19
353	+ 14	403	+ 20	453	+ 32
354	− 3,3	404	+ 1	454	
355	+ 20	405	+ 1	455	
356	+ 24	406	+ 1,8	456	+ 1
357	+ 50	407	+ 3	457	+ 5
358	+ 26	408		458	
359	+ 0,7	409	− 5	459	− 1
360	− 12	410	+ 1	460	+ 1
361	− 5	411	+ 2	461	+ 5
362	+ 5	412	+ 9	462	− 3
363	+ 3,7	413	− 1	463	− 1
364	− 5	414	+ 1,3	464	− 1
365	− 5,7	415	− 1	465	+ 1
366	+ 1	416		466	
367	− 4	417	+ 0,4	467	+ 1
368	− 9	418	− 1	468	+ 1
369	+ 1	419	+ 33	469	+ 1
370	− 26	420	+ 9	470	− 1
371	− 10	421	− 3	471	+ 1
372	+ 4	422	− 6	472	+ 1
373	+ 22	423	+ 1	473	− 1
374	− 1	424	+ 5	474	+ 1
375	− 6	425		475	− 1
376	+ 11	426	− 14	476	+ 1
377	− 89	427	+ 6	477	+ 1
378	+ 13	428	+ 14	478	+ 1
379		429	+ 18	479	
380	− 3	430	+ 19	480	+ 1
381	+ 0,4	431	− 0,2		
382	+ 34	432	− 3		
383	+ 1	433	+ 1,4		
384	− 0,39	434			
385		435	− 0,04		
386	− 3	436	+ 7		
387	+ 2	437	− 3		
388	− 65	438	− 1		
389	− 3	439	+ 2		
390	− 24	440	+ 1		
391	− 2	441	+ 4		
392	− 2	442			
393	− 2	443	− 13		
394	+ 3	444	− 22		
395	− 18	445	− 3		
396	+ 21	446	+ 9		
397		447			
398	− 1	448	+ 9		
399	− 2	449	+ 24		
400	+ 6	450	+ 12		

The comparison of the Orbital and Gravitational Values is unsatisfactory, and appears to show that the Assumed Theory (which has been used only as a basis for corrections that are to be inferred from the Theory now in hand) is sensibly incorrect. The disagreement as regards the forces Normal to the Ecliptic Plane (the subject of Equation (12), Column 77) is usually small; in one instance only does it exceed 2″, the corresponding correction to latitude is below 1″. The disagreement as regards the Transversal Forces in the Plane of the Ecliptic (the subject of Equation (11) Column 75) is large, amounting in one instance to about 30″, and this is exceeded by the disagreement as regards the Ecliptic Radial Forces (Equation (10), Column 74) where the largest discordance is equivalent to an arc of about 50″. It appears probable that the principal parts of these apparent faults arise from imperfections in the assumed expression for Parallax.

We must now prepare for the correction of these by symbolical expressions of the corrections to Orbital and Gravitational values in terms of symbolical corrections of the Fundamental Elements and the Co-efficients of Arguments.

NUMERICAL LUNAR THEORY

SECTION VII

SYMBOLICAL VARIATIONS OF THE THREE FUNDAMENTAL EQUATIONS

PRODUCED BY

SYMBOLICAL VARIATIONS FOR ASSUMED VALUES OF $\frac{a}{r}$, v, AND l;

WITH

THE FIRST FACTORIAL TABLE.

NUMERICAL LUNAR THEORY

SECTION VII.—SYMBOLICAL VARIATIONS OF THE THREE FUNDAMENTAL EQUATIONS, PRODUCED BY SYMBOLICAL VARIATIONS OF THE ASSUMED ELEMENTS.

The numbers in Columns 74, 75, 77, of the three subdivisions of Section VI, represent the amount of apparent failure of each of the Equations (10), (11), (12) for every different argument. It is to be observed that this failure does not originate in error of physical assumption or in failure of the character of the form assumed for satisfying the physical assumption. It is certain (as has been remarked in Section II) that, by arbitrary assumption of those elements which are truly arbitrary (mean distance, ellipticity, and inclination), and applying simple numerical alterations to the other co-efficients and to the periods of fundamental arguments (perigeal and nodal movements), the equations may be perfectly satisfied. It is our object now to form the equations which will lead to the numerical values of those alterations.

It is supposed, for the present, that the Assumed Numerical Values (which are those given as final by Delaunay) are so very nearly accurate that it will be unnecessary to consider the squares or higher powers of the corrections which they may seem to require, and every step will therefore be limited to the first power of those corrections.

On comparing the Tables of Section VI with the Equations (10), (11), (12) of Section I, it will be seen that the resulting numbers in Columns 74, 75, 77, are in reality the numerical values of the following expressions (the terms P, T, Z, which represent occasional disturbing forces, being for the present omitted):—

From Equation (10), Value of Column 74 =

$$+\frac{1}{2}\frac{d}{dt}\left\{\left(\frac{r}{a}\cos l\right)^2\right\}-\left\{\frac{d}{dt}\left(\frac{r}{a}\cos l\right)\right\}^2-\left(\frac{r}{a}\cos l\right)^2\times\left(\frac{dv}{dt}\right)^2+M\frac{a}{r}(\cos l)^2$$
$$-00280\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\cos l\right)^2-00840\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\cos l\right)^2\cos\overline{2(v-V)}$$

From Equation (12), Value of Column 75 =

$$+\frac{d}{dt}\left\{\left(\frac{r}{a}\cos l\right)^2\frac{dv}{dt}\right\}+00839\left(\frac{A}{R}\right)^3\left(\frac{r}{a}\cos l\right)^2\sin\overline{2(v-V)}.$$

From Equation (12), Value of Column 77 =

$$+\frac{d^2}{dt^2}\left(\frac{r}{a}\sin l\right)+M\left(\frac{a}{r}\right)^2\sin l+00560\left(\frac{A}{R}\right)^3\frac{r}{a}\sin l$$

With some smaller terms, whose effects are insensible.

And these are the functions whose numerical values we are so to modify that the new numbers will be those which would be produced by substitution of $\frac{a}{r}+\delta\frac{a}{r}$, $v+\delta v$, $l+\delta l$, instead of $\frac{a}{r}$, v, l, with the view of ultimately introducing numerical values of $\delta\frac{a}{r}$, δv, δl, whose substitution will neutralize the outstanding numerical values of Equations (10), (11), (12). The process is one of simple differentiation.

Each of the factors, of which we are now treating, affects every term in the whole assemblage constituting each of the expressions for (10), (11), (12) respectively Thus we have to form,

$$\frac{d}{dv}\Big(\text{assemblage of terms in (10)}\Big) \times \delta v\,,$$
$$\frac{d}{dv}\Big(\text{assemblage of terms in (11)}\Big) \times \delta v\,,$$
$$\frac{d}{dv}\Big(\text{assemblage of terms in (12)}\Big) \times \delta v\,,$$

and similarly for $\delta\frac{a}{r}$ and δl

We shall hereafter treat of the variation of each individual term in those assemblages

We proceed now with the investigation of the factors in each assemblage of terms

In the following investigations it will frequently be convenient to put the symbol p for $\frac{a}{r}$

The Roman capitals I, II, III in the margin refer to the Equations (10), (11), (12) The numerals attached to them relate to the successive sub-terms of each equation The numerals at the head of each differential line refer to the lines of the following Table of "Factors of Variations as first collected" The order of operations is, that the symbol p is everywhere to be changed to $p + \delta p$, l to $l + \delta l$, and $\frac{dv}{dt}$ to $\frac{dv}{dt} + \delta\frac{dv}{dt}$, and the variations of complex terms are to be made by the same rules as for differentials If there is a sign of differentiation, external to a bracket within which variations are to be performed, the order of these operations is theoretically unimportant, but practically it will be convenient to perform the differentiation last of all

(I) The First Term (for Equation (10), Ecliptic Radial Force) consists of six sub-terms

(I 1) The first sub term is $+\frac{1}{2}\ \frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2 (\cos l)^2\right\}$, or $+\frac{1}{2}\ \frac{d^2}{dt}\{p^{-2}\ (\cos l)^2\}$

The variation of the term under the bracket is—

$$-\,2p^{-3}\ (\cos l)^2\ \delta p - 2p^{-2}\ \cos l\ \sin l\ \delta l\,,$$

and from this we obtain, by double differentiation,

(1) $-\left(\frac{r}{a}\right)^3\ (\cos l)^2$	$\times\ \frac{d^2}{dt^2}\left(\delta\frac{a}{r}\right)$
(2) $-\,2\ \frac{d}{dt}\left\{\left(\frac{r}{a}\right)^3\ (\cos l)^2\right\}$	$\times\ \frac{d}{dt}\left(\delta\frac{a}{r}\right)$
(3) $-\,\frac{d^2}{dt^2}\left\{\left(\frac{r}{a}\right)^3\ (\cos l)^2\right\}$	$\times\ \ \delta\frac{a}{r}$
(4) $\cos l\ \sin l$ (auxiliary)	
(5) $-\left(\frac{r}{a}\right)^2\ \cos l\ \sin l$	$\times\ \frac{d^2}{dt^2}\ \delta l$
(6) $-\,2\,\frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2\ \cos l\ \sin l\right\}$	$\times\ \frac{d}{dt}\ \delta l$
(7) $-\,\frac{d}{dt^2}\left\{\left(\frac{r}{a}\right)^2\ \cos l\ \sin l\right\}$	$\times\ \ \delta l$

The quantities to which the numerals (1), (4), (5) refer are to be formed by combination of some of the developments in the columns of Sections II and III, and their differential co-efficients (2), (3), (6), (7) are formed from them by changing "sine" to "cosine," or "cosine" to "− sine," and multiplying by the "Movement of Argument" exhibited in the

same line which contains the co efficient for the term thus treated The developments of cos l and sin l will also be found in those columns

(I 2) The second sub term of the First Term is $-\left\{\frac{d}{dt}(p^{-1}\cos l)\right\}^2$ Its Variation is,

$$-2\frac{d}{dt}(p^{-1}\cos l)\times\text{Variation of }\frac{d}{dt}(p^{-1}\cos l)$$

$$=-2\frac{d}{dt}(p^{-1}\cos l)\times\frac{d}{dt}\left\{\text{Variation of }(p^{-1}\cos l)\right\}$$

$$=-2\frac{d}{dt}(p^{-1}\cos l)\times\frac{d}{dt}\left\{-p^{-2}\cos l\ \delta p-p^{-1}\sin l\ \delta l\right\}$$

Differentiating the quantity in the large bracket, without separating p^{-2} from cos l, or p^{-1} from sin l, we have

(8) $\frac{r}{a}\cos l$ (auxiliary)

(14) $+2\times$ (9) $\frac{d}{dt}\left(\frac{r}{a}\cos l\right)$	$\times$	(10) $\left(\frac{r}{a}\right)^2\cos l$	$\times\ \frac{d}{dt}\left(\delta\frac{a}{r}\right)$
(15) $+2\times$ (9) $\frac{d}{dt}\left(\frac{r}{a}\cos l\right)$	$\times$	(11) $\frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2\cos l\right\}$	$\times\ \delta\frac{a}{r}$
(16) $+2\times$ (9) $\frac{d}{dt}\left(\frac{r}{a}\cos l\right)$	$\times$	(12) $\frac{r}{a}\sin l$	$\times\ \frac{d}{dt}\delta l$
(17) $+2\times$ (9) $\frac{d}{dt}\left(\frac{r}{a}\cos l\right)$	$\times$	(13) $\frac{d}{dt}\left(\frac{r}{a}\sin l\right)$	$\times\ \delta l$

(I 3) The third sub term of the First Term is $-p^{-2}\cos^2 l\left(\frac{dv}{dt}\right)^2$ Its Variation is

$$+2p^{-3}\cos^2 l\left(\frac{dv}{dt}\right)^2\delta p+2p^{-2}\cos l\sin l\left(\frac{dv}{dt}\right)\delta l-2p^{-2}\cos^2 l\frac{dv}{dt}\frac{d}{dt}\delta v,$$

Or (18) $+2\left(\frac{r}{a}\right)^3\cos^2 l\left(\frac{dv}{dt}\right)^2 \qquad\times\ \delta\frac{a}{r}$

(19) $+2\left(\frac{r}{a}\right)^2\cos l\sin l\left(\frac{dv}{dt}\right)^2 \qquad\times\ \delta l$

(20) $-2\left(\frac{r}{a}\right)^2\cos^2 l\frac{dv}{dt} \qquad\times\ \frac{d}{dt}\delta v$

(I 4) The fourth sub-term of the First Term is $+M\ p\cos^2 l$, where $M=1{\cdot}0027250+\delta M$
Its Variation is

$$+M\cos^2 l\ \delta p-2M\ p\cos l\sin l\ \delta l+p\cos^2 l\ \delta M$$

Or (21) $+1{\cdot}00273\ \cos^2 l \qquad\times\ \delta\frac{a}{r}$

(22) $-2{\cdot}00545\ \frac{a}{r}\cos l\sin l \qquad\times\ \delta l$

(23) $+\frac{a}{r}\cos^2 l \qquad\times\ \delta M$

(I 5) The fifth sub-term is $- 00280 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right) \cos l$ Its Variation is

(24) $+ 00560 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^3 \cos^2 l \qquad \times \delta\frac{a}{r}$

(25) $+ 00560 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right) \cos l \sin l \qquad \times \delta l$

(I 6) The sixth sub-term is $- 00839 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 \cos^2 l \cos \overline{|2(v-V)|}$ Its Variation is

(26) $+ 01679 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^3 \cos^2 l \cos \overline{|2(v-V)|} \qquad \times \delta\frac{a}{r}$

(27) $+ 01679 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 \cos l \sin l \cos \overline{|2(v-V)|} \qquad \times \delta l$

(28) $+ 01679 \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 \cos^2 l \sin \overline{|2(v-V)|} \qquad \times \delta v$

These six sub-terms, (I 1), (I 2), (I 3), (I 4), (I 5), (I 6), must be united to form the Variation of Equation (10)

(II) The Second Term (or that for Transversal Ecliptic Forces) consists of two sub-terms

(II 1) The first sub-term is $+\frac{d}{dt}\left\{\left(\frac{r}{a} \cos l\right) \frac{dv}{dt}\right\}$ or $+\frac{d}{dt}\left\{p^{-2} (\cos l) \frac{dv}{dt}\right\}$ Its Variation is

(29) $- 2 \left(\frac{r}{a}\right)^3 (\cos l)^2 \frac{dv}{dt} \qquad \times \frac{d}{dt}\left(\delta\frac{a}{r}\right)$

(30) $- 2 \frac{d}{dt}\left\{\left(\frac{r}{a}\right)^3 (\cos l)^2 \frac{dv}{dt}\right\} \qquad \times \delta\frac{a}{r}$

(31) $- 2 \left(\frac{r}{a}\right)^2 \cos l \sin l \frac{dv}{dt} \qquad \times \frac{d}{dt}(\delta l)$

(32) $- 2 \frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2 \cos l \sin l \frac{dv}{dt}\right\} \qquad \times \delta l$

(33) $+ \left(\frac{r}{a}\right)^2 (\cos l)^2 \qquad \times \frac{d}{dt}\delta v$

(34) $+ \frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2 (\cos l)^2\right\} \qquad \times \frac{d}{dt} v$

(II 2) The second sub-term is $+ 00839 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 (\cos l)^2 \sin \overline{|2(v-V)|}$ Its Variation is

(35) $- 01679 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^3 (\cos l)^2 \sin \overline{|2(v-V)|} \qquad \times \delta\left(\frac{a}{r}\right)$

(36) $- 01679 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 \cos l \sin l \sin \overline{|2(v-V)|} \qquad \times \delta l$

(37) $+ 01679 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2 (\cos l)^2 \cos \overline{|2(v-V)|} \qquad \times \delta v$

The developments, whose combinations are required to form these quantities, will be found partly in Section II, partly in Section IV The sub terms (II 1) and (II 2) must be united to form the Variation of Equation (11)

(III) The Third Term (for Forces Normal to the Ecliptic) consists of three sub-terms

(III 1) The first sub term is $+\frac{d^2}{dt^2}\left(\frac{r}{a}\sin l\right)$ The Variation of the term under the bracket is

$$-p^{-2}\sin l\ \delta p + p^{-1}\cos l\ \delta l$$

$$\text{Or } -\left\{\left(\frac{r}{a}\right)^2\sin l\right\}\delta p + \frac{r}{a}\cos l\ \delta l$$

Its second differential is

$$(38)\quad -\left(\frac{r}{a}\right)^2\sin l \qquad\times\ \frac{d}{dt}\left(\delta\frac{a}{r}\right)$$

$$(39)\quad -2\frac{d}{dt}\left\{\left(\frac{r}{a}\right)^2\sin l\right\} \qquad\times\ \frac{d}{dt}\left(\delta\frac{a}{r}\right)$$

$$(40)\quad -\frac{d^2}{dt}\left\{\left(\frac{r}{a}\right)^2\sin l\right\} \qquad\times\ \delta\frac{a}{r}$$

$$(41)\quad +\frac{r}{a}\cos l \qquad\times\ \frac{d}{dt^2}\delta l$$

$$(42)\quad +2\frac{d}{dt}\left(\frac{r}{a}\cos l\right) \qquad\times\ \frac{d}{dt}\delta l$$

$$(43)\quad +\frac{d^2}{dt^2}\left(\frac{r}{a}\cos l\right) \qquad\times\ \delta l$$

(III 2) The second sub-term is $+M\ p^2\sin l$ Its Variation is

$$+2M\ p\ \sin l\ \delta p + M\ p^2\cos l\ \delta l + p^2\sin l\ \delta M$$

$$\text{Or } (44)\quad +2.00545\ \frac{a}{r}\sin l \qquad\times\ \delta\left(\frac{a}{r}\right)$$

$$(45)\quad +1.00273\ \left(\frac{a}{r}\right)^2\cos l \qquad\times\ \delta l$$

$$(46)\quad +\left(\frac{a}{r}\right)^2\sin l \qquad\times\ \delta M$$

(III 3) The third sub-term is $+.00560\ \left(\frac{A}{R}\right)^3 p^{-1}\sin l$ Its Variation is

$$(47)\quad -.00560\ \left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)^2\sin l \qquad\times\ \delta\frac{a}{r}$$

$$(48)\quad +.00560\ \left(\frac{A}{R}\right)^3\left(\frac{r}{a}\right)\cos l \qquad\times\ \delta l$$

The three sub terms (III 1), (III 2), (III 3) must be united to form the complete Variation of Equation (12)

I now proceed to collect all the terms of these Variations in a Table, exhibiting the form of each factor of an elementary Variation, and the process by which its numerical development has been obtained

When the algebraic expressions for the factors could be found in the formulæ at the heads of the columns of Section II and Section IV, the numbers were extracted from those columns to the 4th decimal For other terms, whose expressions do not find place in Sections II and IV, it was found possible to combine formulæ of different columns in those Sections, so as to produce the expression required Attention was given to ensure the accuracy of the constant terms (the first column in the table) to the sixth decimal

Calculations were completed for the following arguments —0, l, $2D-l$, $2D$, $2l$, $2D+l$, $2D-S$, $2D-l-S$, $l-S$, D, $l+S$, $2f-l$, S, $2D-2l$, f, $f+l$, $f-l$, $2D-f$, $2D+f-l$, $2D+f$ $f-2l$ But, for the greater portion of these terms, the numbers, after assemblage for the Factorial Table, became so small that it was evidently useless to retain them in the first Table Finally I excluded all whose value does not rise to 005

The different serial multipliers for the same Variation were then collected, and the results were arranged in the more convenient form of the First Factorial Table

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NUMERICAL LUNAR THEORY

SERIAL FACTORS AS FIRST COLLECTED FROM THE

	T g m lF l f F	f Alg l E p	E C m l	C Sl l P d	COEFFICIENTS IN THE PRODUCTS OF FACTORS							
FOR EQUATION ( )												
( )	( ) ( )	0 1 7 0 1 4		0	53	6 7	7/	3				
( )	$\frac{d}{d}\{( )$ (l) }	$\frac{d}{d}$( )		b		3 8	476	79	l		8	6
( )	$\frac{d}{d}\{( )$ (D }	$\frac{d}{d}$( )		0		6	5	78	4	–	7	6
( )	(l y)	0 1 6 0 1		Sl								
(5)	( ) l l	(4) 0 1		S								
(6)	$\frac{d}{d}\{( )$ l l }	$\frac{d}{d}$(5)		0								
(7)	$\frac{d}{d}\{( )$ l }	$\frac{d}{d}$(6)		S								
(8)	l(l ry)	0 1 3		0	999545	545	96	75				00
(9)	$\frac{d}{d}\{$ os l } (h y)	$\frac{d}{d}$( )		9		539	8	39				– 004
( )	( ) l(l y)	0 1 7 0 1 3		0	663	88	88	6		–		008
( )	$\frac{d}{d}\{( )$ l } (lary)	$\frac{d}{d}$( )		Sl		79	6	7			4	– 004
( )	l(l ry)	0 1 8		Sl								
( )	$\frac{d}{d}\{$ l } (xl ar)	$\frac{d}{d}$( )		C								
( )	$\frac{d}{d}$(l) ( ) l	(9) ( )		5		8	56	6			1	008
(5)	$\frac{d}{d}$(l) $\frac{d}{d}\{( )$ l }	(9) ( )		0 1	635	8	8	6	(			
(6)	$\frac{d}{d}$(l) l	(9) ( )		C								
(7)	$\frac{}{d}$(l) $\frac{d}{d}\{$ l }	(9) (5)		S								
(8)	( ) (D $\left(\frac{d}{d}\right)$	0 1 7 0 1 9		C	985686	7	14	338	6			– 0 8
(9)	( ) l l$\left(\frac{d}{d}\right)$	(4) 0 1 6		5								
( )	( ) (co D $\frac{d}{d}$	0 1 7		0	988789	6	3	9				
( )	(D	0 1		0	998688							
( )	l	0 1 (4)	54	S								
(3)	(D	0 1 0 1		0	995975	543	99	8	3			004
(4)	$\left(\frac{A}{R}\right)$ ( ) (D	$\left\{\begin{matrix}0\ 1\ 3 & 0\ 1\ 7\\ 0\ 1 & \end{matrix}\right\}$	00560	0	56 4	9						
(5)	$\left(\frac{A}{R}\right)$ ( ) l l	0 1 33 (4)	5	S								
(6)	$\left(\frac{A}{R}\right)$ ( ) (D $\overline{(V}$	$\left\{\begin{matrix}0\ 1\ 33 & 0\ 1\ 7\\ 0\ 1 & 0\ 1\ 56\end{matrix}\right\}$	679	C	8	5	3	68				
(7)	$\left(\frac{A}{R}\right)$ ( ) l l l or $\overline{(V)}$	0 1 33 (4) 0 1 56	679	C								
(8)	$\left(\frac{A}{R}\right)$ ( ) (D n $\overline{(V)}$	$\left\{\begin{matrix}0\ 1\ 33 & 0\ 1\\ 0\ 1\ 55 & \end{matrix}\right\}$	79	5		5	9	68				

Section VII.—Symbolical Variations of Fundamental Equations

Several Terms of the Three Equations

By Terms of Section II					Fundamental Variations										No. for Reference
Terms formed from Section II., Order B.					Each to be multiplied by one Co-efficient in the same line.										
f	$[f\ i]$	$[f-i]$	$[D\ f]$	$[aD\ f\ a]$		$\frac{d}{dt}(s)$	$\frac{d^2}{dt^2}(s)$	δv	$\frac{d}{dt}\delta v$	$\frac{d^2}{dt^2}\delta v$	M	$\frac{d}{dt}M$	$\frac{d^2}{dt^2}M$	δM	
For Equation ( )															
							$\frac{d^2}{dt^2}(s)$								( )
						$\frac{d}{dt}(s)$				[illegible]					( )
					s										(3)
– 089	005	005	003	00											(4)
– 89		6	– 004	– 00									$\frac{d^2}{dt^2}M$		(5)
– 78			006	– 004								$\frac{d}{dt}M$			(6)
+ 89			003	003							M				(7)
															(8)
															(9)
															( )
															( )
090	002	– 007	003	00											( )
090	004		003	90											(3)
						$\frac{d}{dt}(s)$									(14)
															(5)
	004	– 006	002									$\frac{d}{dt}M$			(6)
	+ 004		002								M				(7)
					s										(8)
78	080		004	002							M				(9)
									$\frac{d}{dt}\delta v$						(20)
			+ 004		s										( )
– 178	– 4	006	006	002							M				( )
														δM	(23)
					s										(24)
+ 00											M				(5)
					s										(6)
			00								M				(7)
								δv							(28)

SERIAL FACTORS AS FIRST COLLECTED FROM THE

A	g lF f F	M l f Alg b F	E l C fh	C 6 P d	Coefficients in the products of factor: T m f d f				H O 1 A			
						l	[l A	l l	l l	lu	l	l f
FOR EQUATION ( )												
( )	$() \; (\; 1) \frac{d}{d}$	C C 1		C	99 9	88		(				
( )	$\frac{l}{d}\{() \; (\; 1) \frac{l}{d}\}$	$\frac{l}{dt}$(9)		6		79	9		4			
( )	$() \quad 1 \quad 1 \frac{d}{d}\}$	C 1 6(4) C 1 6		6								
(3)	$\frac{l}{l}\{() \quad 1 \quad 1 \frac{d}{d}\}$	$\frac{l}{d}$(3)		C								
(33)	$() \; (\; 1)$	C 4		C	64	84	88	45	—			4
(3)	$\frac{l}{}\{() \; (\;)\}$	$\frac{d}{l}$(3)		3		75	(	68			—	8
(5)	$(l) \; () \; (\; 1) \quad \overline{| \; (\; V)|}$	{C 1 3 C 1 C 7 C 1 55}	9	8		5	34	68		—		
(3)	$(\frac{A}{l}) \; () \quad 1 \quad \overline{| \; (\; V)|}$	C 1 33 ( ) C 55	6 9	C								
(37)	$(\frac{A}{R}) \; () \; (\; 1) \quad \overline{| \; (\; V)|}$	{C 1 3 C 1 C 1 56}	6 9	C	36	5	7	(8				
FOR EQUATION ( )												
(38)	$()$	C 1 6 C 1 7		9								
(39)	$\frac{l}{dt}\{() \quad 1\}$	$\frac{d}{d}$(38)		C								
(4)	$\frac{d}{d}\{() \quad \}$	$\frac{d}{d}$(39)		3								
(4)	1	(8)		C	999545	543	96	75	—			
(4)	$\frac{d}{d} \; (\quad 1)$	$\frac{d}{d}$(4)		8		78	64	78	4	4		
(43)	$\frac{l}{d} \; (\quad 1)$	$\frac{d}{d}$(4)		C		535	7	38	4	4	—	8
(44)	1 1	C C 1 7	545	8								
(45)	$() \quad 1$	C 1 8 C 1	73	C	79	94	5	7	7			
(46)	$() \; 1 \; 1$	C 1 8 C 1 7		8								
(47)	$(\frac{A}{R}) \; () \quad 1$	C 1 C 1 7	56	8								
(48)	$(\frac{A}{R}) \quad 1$	{C 1 33 C 1 C 1}	56	C	55 7	3						

SECTION VII.—SYMBOLICAL FUNDAMENTAL EQUATIONS—*continued*

[S]EVERAL TERMS OF THE THREE EQUATIONS.

[B]Y TERMS OF SECTION II.					FUNDAMENTAL VARIATIONS.										No. for Reference.
Terms formed from Section II., Order B.					Each to be multiplied by one Co-efficient in the same line.										
f	$\overline{f+l}$	$\overline{f-l}$	$\overline{D-f}$	$\overline{2D\ f-l}$	s	$\frac{d}{dt}(s)$	$\frac{d^2}{dt^2}(s)$	δw	$\frac{d}{dt}\delta w$	$\frac{d^2}{dt^2}$	n	$\frac{d}{dt}n$	$\frac{d^2}{dt^2}n$	δM	
FOR EQUATION ( ).															
						$\frac{d}{dt}(s)$									(29)
					s										(30)
− 78			− 006	002								$\frac{d}{dt}n$			(31)
8	020		− 006	004							n				(32)
										$\frac{d^2}{dt^2}$					(33)
									$\frac{d}{dt}\delta w$						(34)
					s										(35)
			− 001								n				(36)
								s							(37)
FOR EQUATION ( ).															
089			− 05	− 00			$\frac{d^2}{dt^2}(s)$								(38)
78			− 006	− 004		$\frac{d}{dt}(s)$									(39)
89			1	003	s										(40)
													$\frac{d^2}{dt^2}n$		(41)
												$\frac{d}{dt}n$			(42)
											n				(43)
78	4	− 006	006	002	s										(44)
											n				(45)
089	+		02	00										δM	(46)
00					s										(47)
											n				(48)

FACTORS OF VARIATIONS FOR

FOR EQUATION (10)

δ Equation (10) =

$+\delta\left(\frac{a}{r}\right)$	$\times$	$\{ + 2\ 996059$	$-\ 0537 \cos l$	$-\ 0069 \cos \overline{2D-l}$
$+\frac{d}{dt}\left(\delta\frac{a}{r}\right)$	$\times$	$\{$	$-\ 2142 \sin l$	$-\ 0320 \sin \overline{2D-l}$
$+\frac{d^2}{dt^2}\left(\delta\frac{a}{r}\right)$	$\times$	$\{ - 1\ 005301$	$+\ 1627 \cos l$	$+\ 0277 \cos \overline{2D-l}$
$+\delta v$	$\times$	$\{$	$-\ 0005 \sin l$	$-\ 0029 \sin \overline{2D-l}$
$+\frac{d}{dt}\delta v$	$\times$	$\{ - 1\ 988789$	$+\ 0006 \cos l$	$+\ 0030 \cos \overline{2D-l}$
$+\delta M$	$\times$	$\{ +\ \ 995975$	$+\ 0543 \cos l$	$+\ 0099 \cos \overline{2D-l}$

$$\left\{+\frac{1}{a}\frac{d}{dr}\left(Pr\cos 1\right)\times\delta r+\frac{1}{a}\frac{d}{dv}\left(Pr\cos 1\right)\times\delta v+\frac{1}{a}\frac{d}{d1}\left(Pr\cos 1\right)\times\delta 1\right\}\times\left\{+\ 995545\right.$$

$+\delta 1$	$\times$	$\{$	$+\ 090 \sin f$	$+\ 010 \sin \overline{f+l}\ \}$
$+\frac{d}{dt}\delta 1$	$\times$	$\{$	$-\ 178 \cos f$	
$+\frac{d^2}{dt^2}\delta 1$	$\times$	$\{$	$-\ 089 \sin f$	

FOR EQUATION (11)

δ Equation (11) =

$+\delta\left(\frac{a}{r}\right)$	$\times$	$\{$	$-\ 1074 \sin l$	$-\ 0157 \sin \overline{2D-l}$
$+\frac{d}{dt}\left(\delta\frac{a}{r}\right)$	$\times$	$\{ - 1\ 991900$	$+\ 1080 \cos l$	$+\ 0222 \cos \overline{2D-l}$
$+\delta v$	$\times$	$\{ -\ \ 000236$	$-\ 0005 \cos l$	$-\ 0027 \cos \overline{2D-l}$
$+\frac{d}{dt}\delta v$	$\times$	$\{$	$+\ 1075 \sin l$	$+\ 0161 \sin \overline{2D-l}$
$+\frac{d}{dt}\delta v$	$\times$	$\{ + 1\ 000642$	$-\ 1084 \cos l$	$-\ 0188 \cos \overline{2D-l}$

$$\left\{+\frac{1}{a}\frac{d}{dr}\left(Tr\cos 1\right)\times\delta r+\frac{1}{a}\frac{d}{dv}\left(Tr\cos 1\right)\times\delta v+\frac{1}{a}\frac{d}{d1}\left(Tr\cos 1\right)\times\delta 1\right\}\times\left\{+\ 995545\right.$$

$+\delta 1$	$\times$	$\{$	$-\ 178 \cos f$	$-\ 020 \cos \overline{f+l}$
$+\frac{d}{dt}\delta 1$	$\times$	$\{$	$-\ 178 \sin f$	$-\ 010 \sin \overline{f+l}$

FOR EQUATION (12)

δ Equation (12) =

$+\delta\left(\frac{a}{r}\right)$	$\times$	$\{$	$+\ 266 \sin f$	$+\ 014 \sin \overline{f+l}$
$+\frac{d}{dt}\left(\delta\frac{a}{r}\right)$	$\times$	$\{$	$-\ 178 \cos f$	
$+\frac{d}{dt^2}\left(\delta\frac{a}{r}\right)$	$\times$	$\{$	$-\ 089 \sin f$	
$+\delta M$	$\times$	$\{$	$+\ 089 \sin f$	$+\ 010 \sin \overline{f+l}\ \}$

$$\left\{\frac{1}{a}\frac{dZ}{dr}\times\delta r+\frac{1}{a}\frac{dZ}{dv}\times\delta v+\frac{1}{a}\frac{dZ}{d1}\times\delta 1\right\}\times\left\{+\ 1\ 000000\right\}$$

$+\delta 1$	$\times$	$\{ + 1\ 007856$	$+\ 1626 \cos l$	$+\ 0274 \cos \overline{2D-l}$
$+\frac{d}{dt}\delta 1$	$\times$	$\{$	$+\ 1078 \sin l$	$+\ 0164 \sin \overline{2D-l}$
$+\frac{d^2}{dt^2}\delta 1$	$\times$	$\{ -\ \ 999545$	$-\ 0543 \cos l$	$-\ 0096 \cos \overline{2D-l}$

COMPLETION OF FUNDAMENTAL EQUATIONS

FOR EQUATION (10)	No for Reference
$-\ 0240 \cos \lvert\overline{2D}\rvert \quad +\ 005 \cos \lvert\overline{2l}\rvert \quad +\ 008 \cos \lvert\overline{2f}\rvert \}$	[1]
$-\ 0530 \sin \lvert\overline{2D}\rvert \quad +\ 008 \sin \lvert\overline{2f}\rvert \}$	[2]
$+\ 0213 \cos \lvert\overline{2D}\rvert \}$	[3]
$+\ 0168 \sin \lvert\overline{2D}\rvert \}$	[4]
$-\ 0090 \cos \lvert\overline{2D}\rvert \}$	[5]
$+\ 0082 \cos \lvert\overline{2D}\rvert \}$	[6]
$-\ 0543 \cos l \quad -\ 0096 \cos \lvert\overline{2D-l}\rvert \quad -\ 0075 \cos \lvert\overline{2D}\rvert \}$	[7]
	[8]
	[9]
$+\ 010 \sin \lvert\overline{f-l}\rvert \}$	[10]

FOR EQUATION (11)	No for Reference
$-\ 0279 \sin \lvert\overline{2D}\rvert \}$	[11]
$+\ 0060 \cos \lvert\overline{2D}\rvert \}$	[12]
$+\ 0168 \cos \lvert\overline{2D}\rvert \}$	[13]
$+\ 0268 \sin \lvert\overline{2D}\rvert \quad -\ 008 \sin \lvert\overline{2f}\rvert \}$	[14]
$-\ 0145 \cos \lvert\overline{2D}\rvert \quad +\ 004 \cos \lvert\overline{2f}\rvert \}$	[15]
$-\ 0543 \cos l \quad -\ 0096 \cos \lvert\overline{2D-l}\rvert \quad +\ 0075 \cos \lvert\overline{2D}\rvert \}$	[16]
$-\ 007 \cos \lvert\overline{2D-f}\rvert \}$	[17]
$+\ 010 \sin \lvert\overline{f-l}\rvert \quad -\ 006 \sin \lvert\overline{2D-f}\rvert \}$	[18]

FOR EQUATION (12)	No for Reference
$-\ 006 \sin \lvert\overline{f-l}\rvert \quad +\ 009 \sin \lvert\overline{2D-f}\rvert \quad +\ 005 \sin \lvert\overline{2D+f-l}\rvert \}$	[19]
$-\ 006 \cos \lvert\overline{2D-f}\rvert \}$	[20]
$+\ 010 \sin \lvert\overline{f-l}\rvert \}$	[21]
	[22]
	[23]
$+\ 0427 \cos \lvert\overline{2D}\rvert \quad +\ 011 \cos \lvert\overline{2l}\rvert \quad +\ 005 \cos \lvert\overline{2D-S}\rvert \quad -\ 006 \cos \lvert\overline{2f}\rvert \}$	[24]
$+\ 0278 \sin \lvert\overline{2D}\rvert \}$	[25]
$-\ 0075 \cos \lvert\overline{2D}\rvert \}$	[26]

NUMERICAL LUNAR THEORY

SECTION VIII.

INTRODUCTION OF NEW NOTATION

AND FORMATION OF

MODIFIED FACTORIAL TABLE

WITH STEPS FOR

DETAILED FINAL EQUATIONS OF CORRECTIONS TO ASSUMED CO-EFFICIENTS

SECTION VIII —INTRODUCTION OF NEW NOTATION AND FORMATION OF MODIFIED FACTORIAL TABLE WITH STEPS FOR DETAILED FINAL EQUATIONS OF CORRECTIONS TO ASSUMED CO-EFFICIENTS

We may now describe the New Notation which it has been found convenient occasionally to employ

Every horizontal line in the table of Section II, Part 2, exhibits multipliers of cosine or sine, each of a single argument, the successive values of that argument being 0, l, $\overline{2D-l}$, $\overline{2D}$, $\overline{2l}$, &c We shall refer to these arguments by the general letter H, with subscript for each argument, the same as the "No for Reference," thus for 0, l, $\overline{2D-l}$, $\overline{2D}$, $\overline{2l}$, &c, we shall use H_1, H_2, H_3, H_4, H_5, &c And, for their separate numerical co efficients, in Section II, Part 2, Column No 1, we shall use the general letter g with the same special subscript, and in Section II Part 2, Column No 15, we shall use the letter h with the same special subscript, and similarly for Movements of Arguments we shall use the letter m with the same special subscript Thus—

$g_2 \cos H_2 = +545095 \cos l,\quad h_2 \sin H_2 = +1097572 \sin l,\quad m_2 = +0\ 9915480$

$g_3 \cos H_3 = +99813 \cos \overline{2D-l},\quad h_3 \sin H_3 = +222336 \sin \overline{2D-l},\quad m_3 = +0\ 8588494$

&c, &c, &c

A similar system applies to the table of Section II Part 3, the symbols K_{301}, K_{302}, K_{303}, K_{304}, &c being used for f, $\overline{f+l}$, $\overline{f-l}$, $\overline{2D-f}$, &c, and the symbols k_{301}, k_{302}, k_{303}, k_{304}, &c for the co efficients of their sines, and m_{301}, m_{302}, m_{303}, m_{304}, &c for their movements of argument

Now for the formation of that term of δ Equation (10) which depends on $\delta\left(\frac{a}{r}\right)$, we must use $\{+2\ 996059 - 0537 \cos l + \&c\}$ multiplied by $\left\{\text{the sum of all the values which constitute } \delta\frac{a}{r}\right\}$, that is, multiplied by the sum of all the possible values of $\delta g \cos H$ That sum we shall express by $\Sigma(\delta g \cos H)$, in the evaluation of which we are to use all the values $\delta g_1 \cos H_1$, $\delta g_2 \cos H_2$, &c For that term which depends on $\frac{d}{dt}\left(\delta\frac{a}{r}\right)$, we must use $\{-\ 2142 \sin l -, \&c\}$ multiplied by the sum of all the possible values of $-\delta g \sin H\ m\}$

The result of multiplication by $-\ 0537 \cos l$ will be $-\ 0268 \times \{+(\cos l + \cos l)\,\delta g_1$, $+(\cos 0 \cos \overline{2l})\,\delta g + (\cos \overline{2D-2l} + \cos \overline{2D})\,\delta g_3 +, \&c\}$, and all the succeeding terms of development of line [1] will be generally similar to this

The following lines [2], [3], [5,] introduce, besides a change of the left-hand multiplier, a new variable m It will be remarked that $\sin 0 = 0$, and thus, the first term to be considered is—

$$-\ 2142 \sin l \times \{-\sin l\ m_2\ \delta g_2 - \sin \overline{2D-l}\ m_3\ \delta g_3 -, \&c\}$$

$$\text{or} +\ 1071 \times \{(+1 - \cos 2l)\ m_2\ \delta g_2 + (\cos \overline{2D-2l} - \cos \overline{2D})\ m_3\ \delta g_3) +, \&c\}$$

These terms present no difficulty (though sufficiently laborious) By means of properly arranged skeleton forms, they are computed with very little risk of inaccuracy

It is, however, very important to remark that the co efficients g, h, k, do not enter into these formulæ And thus, the variation of the smallest or most distant co-efficient in Column 1, 15, or 24, may receive a multiplier as large as those received by the variations of the early co efficients

NO	TERMS OF THE FACTORIAL TABLE FOR δ EQUATION (10)				
[1]	{ + 2 996059	$- 0037 \cos l$	$- 0069 \cos \overline{2D - l}$	$- 0240 \cos \overline{2D}$	
[2]	{	$- [illegible]142 \sin l$	$- 0320 \sin \overline{2D - l}$	$- 0530 \sin \overline{2D}$	
[3]	{ − 1 005301	$+ 1627 \cos l$	$+ 0277 \cos \overline{2D - l}$	$+ 0213 \cos \overline{2D}$	}
[4]	{	$- 0005 \sin l$	$- 0029 \sin \overline{2D - l}$	$+ 0168 \sin \overline{2D}$	}
[5]	{ − 1 988789	$+ 0006 \cos l$	$+ 0030 \cos \overline{2D - l}$	$- 0090 \cos \overline{2D}$	}
[6]	{ + 995975	$+ 0543 \cos l$	$+ 0099 \cos \overline{2D - l}$	$+ 0082 \cos \overline{2D}$	}
[7]	{ + 995545	$- 0543 \cos l$	$- 0096 \cos \overline{2D - l}$	$- 0075 \cos \overline{2D}$	}
[8]	{	$+ 090 \sin f$	$+ 010 \sin \overline{f - l}$		
[9]	{	$- 178 \cos f$			
[10]	{	$- 089 \sin f$		$+ 010 \sin \overline{f - l}$	}
	TERMS OF THE FACTORIAL TABLE FOR δ EQUATION (11)				
[11]	{	$- 1074 \sin l$	$- 0157 \sin \overline{2D - l}$	$- 0279 \sin \overline{2D}$	}
[12]	{ − 1 991900	$+ 1088 \cos l$	$+ 022[illegible] \cos \overline{2D - l}$	$+ 0060 \cos \overline{2D}$	}
[13]	{ − 000236	$+ 0005 \cos l$	$+ 0027 \cos \overline{2D - l}$	$+ 0166 \cos \overline{2D}$	}
[14]	{	$+ 1075 \sin l$	$+ 0161 \sin \overline{2D - l}$	$+ 0268 \sin \overline{2D}$	
[15]	{ + 1 000642	$- 1084 \cos l$	$- 0188 \cos \overline{2D - l}$	$- 0145 \cos \overline{2D}$	
[16]	{ + 995545	$- 0543 \cos l$	$- 0096 \cos \overline{2D - l}$	$+ 0075 \cos \overline{2D}$	}
[17]	{	$- 178 \cos f$	$- 020 \cos \overline{f + l}$		
[18]	{	$- 178 \sin f$	$- 010 \sin \overline{f + l}$	$+ 010 \sin \overline{f - l}$	
	TERMS OF THE FACTORIAL TABLE FOR δ EQUATION (12)				
[19]	{	$+ 266 \sin f$	$+ 014 \sin \overline{f - l}$	$- 006 \sin \overline{f - l}$	
[20]	{	$- 178 \cos f$			
[21]	{	$- 089 \sin f$		$+ 010 \sin \overline{f - l}$	}
[22]	{	$+ 089 \sin f$	$+ 010 \sin \overline{f - l}$		
[23]	{ + 1 000000				
[24]	{ + 1 007856	$+ 1626 \cos l$	$+ 0274 \cos \overline{2D - l}$	$+ 0427 \cos \overline{2D}$	
[25]	{	$+ 1078 \sin l$	$+ 0164 \sin \overline{2D - l}$	$+ 0278 \sin \overline{2D}$	}
[26]	{ + 999545	$- 0543 \cos l$	$- 0096 \cos \overline{2D - l}$	$- 0075 \cos \overline{2D}$	}

Section VIII —modified factorial table

SYMBOLICAL VARIATIONS OF ASSUMED CO EFFICIENTS				NO
+ 005 cos $\lvert 2l \rvert$	+ 008 cos $\lvert 2f \rvert \}$	×	$\Sigma \{+\cos H\ \delta g\ \}$	[1]
	+ 008 sin $\lvert 2f \rvert \}$	×	$\Sigma \{-\sin \Pi\ m\ \delta g\}$	[2]
		×	$\Sigma \{-\cos \Pi\ m\ \delta g\}$	[3]
		×	$\Sigma \{+\sin \Pi\ \delta h\ \}$	[4]
		×	$\Sigma \{+\cos H\ m\ \delta h\}$	[5]
		×	$+\delta M$	[6]
			$+ P,$ cos 1	[7]
		×	$\Sigma \{+\sin K\ \delta k\ \}$	[8]
		×	$\Sigma \{+\cos K\ m'\ \delta k\}$	[9]
		×	$\Sigma \{-\sin K\ m'^2\ \delta k\}$	[10]

SYMBOLICAL VARIATIONS OF ASSUMED CO EFFICIENTS				
		×	$\Sigma \{+\cos H\ \delta g\ \}$	[11]
		×	$\Sigma \{-\sin \Pi\ m\ \delta g\}$	[12]
		×	$\Sigma \{+\sin \Pi\ \delta h\ \}$	[13]
	− 008 sin $\lvert 2f \rvert \}$	×	$\Sigma \{+\cos \Pi\ m\ \delta h\}$	[14]
	+ 004 cos $\lvert 2f \rvert \}$	×	$\Sigma \{-\sin \Pi\ m\ \delta h\}$	[15]
			$+ I,$ cos 1	[16]
− 007 cos $\lvert 2D - f \rvert \}$		×	$\Sigma \{+\sin K\ \delta k\ \}$	[17]
− 006 sin $\lvert 2D - f \rvert \}$		×	$\Sigma \{+\cos K\ m^1\ \delta k\}$	[18]

SYMBOLICAL VARIATIONS OF ASSUMED CO EFFICIENTS				
+ 009 sin $\lvert 2D - f \rvert$	+ 00? sin $\lvert 2D + f - l \rvert \}$	×	$\Sigma \{+\cos \Pi\ \delta g\ \}$	[19]
− 006 cos $\lvert 2D - f \rvert \}$		×	$\Sigma \{-\sin H\ m\ \delta g\}$	[20]
		×	$\Sigma \{-\cos \Pi\ m^2\ \delta g\}$	[21]
		×	$+\delta M$	[22]
			$+ Z$	[23]
+ 011 cos $\lvert 2l \rvert$	+ 005 cos $\lvert 2D - S \rvert$ − 006 cos $\lvert 2f \rvert$ $\}$	×	$\Sigma \{+\sin K\ \delta k\ \}$	[24]
		×	$\Sigma \{+\cos K\ m'\ \delta k\}$	[25]
		×	$\Sigma \{-\sin K\ m'\ \delta k\}$	[26]

We have now to convert these formulæ into others, which bear directly upon the expression of the variations of δ Equations (10), (11), (12), by means of the individual variations δq_1, δq_2, δq_3, &c., δh_1, δh_2, δh_3, &c., δk_1, δk_2, δk_3, &c., and of the general variation δM, with factors no more complicated than simple sines and cosines of arguments which we have yet to select. It will be remarked that all the brackets on the right hand contain terms multiplying δq, δh, δk.

Giving our attention to line [1], and bearing in mind the import of the sign Σ, it will be seen that the result of multiplication by $+2 \cdot 996059$ will be—

$$+2 \cdot 996059 \times \left\{\cos 0 \ \delta q_1 + \cos l \ \delta q_2 + \cos \overline{|2D-l|} \ \delta q_3 + \&c. \right\}$$

a simple series. It will be remembered that $\cos 0 = 1$.

The actual multiplications, by which the great products of the Modified Factorial Table are effected, are placed in the collection of papers of the Numerical Lunar Theory under the title "Product Sheets," by which we shall, if necessary, further refer to them. For each of the three Equations, one sheet contains all the multiplications corresponding to a single value of H and a single value of K in the "Symbolical Variations of Assumed Co-efficients" of the Modified Factorial Table. As it is proposed to suspend this part of the work after the 100th term, there are 100 sheets for Equation (10), 100 for Equation (11), and 100 for Equation (12). One sheet of Equation (10), contains 19 multiplications of the quantities shewn in the "Lines" of the Factorial Table; one sheet of Equation (11), contains 23 such multiplications; and one of Equation (12), contains 18. The marginal reference indicating the product of the co-efficients is distinguished by the word PRODUCT in small capitals, and we shall use this in the same sense through the following pages. A Specimen of one of each of these sheets is subjoined.

NUMERICAL LUNAR THEORY (I, 10)

$H_3 = \boxed{2D-l}$
$K_{342} = \boxed{f+l}$

$m = +\ 859,\ m^2 = +\ 738$
$m' = +1\ 996,\ m'^2 = +3\ 984$

Numerical Formation of the Co-efficients of δg, δh, δk, for δ Equation (10)

Co-efficients to be formed	[4] for δh_3	[5] for δh_3		[1] for δg_3			[2] for δg_3			
Factorial Trigonometrical Term	sin $\boxed{2D}$	cos O	cos $\boxed{2D}$	cos O	cos l	cos $\boxed{2D}$	sin l	sin $\boxed{2D-l}$	sin $\boxed{2D}$	sin $\boxed{2f}$
Variational Trigonometrical Term	sin $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	sin $\boxed{2D-l}$	sin $\boxed{2D-l}$	sin $\boxed{2D-l}$	sin $\boxed{2D-l}$
Factorial Co-efficient 1, or m, or m^2	+ 017	− 1 989 958 $+m$	− 009 958 $+m$	+ 2 996	− 054	− 024	− 214 958 $-m$	− 032 958 $-m$	− 0 [illegible] 958 $-m$	+ 008 958 $-m$
	(Figures of multiplication)						(Figures of multiplication)			
PRODUCT	+ 017	− 1 708	− 007	+ 2 996	− 054	− 024	+ 182	+ 026	+ 045	− 006
New Co-efficient	− 01	− 85		+ 1 50	− 03	− 01	− 09	− 01	− 02	
New Tr Term with Sum of Arguments	cos $\boxed{4D-l}$	cos $\boxed{2D-l}$	cos	cos $\boxed{2D-l}$	cos $\boxed{2D}$	cos $\boxed{4D-l}$	cos $\boxed{2D}$	cos $\boxed{4D-2l}$	cos $\boxed{4D-l}$	cos
New Co-efficient	+ 01	− 86		+ 1 50	− 03	− 01	+ 09	+ 01	+ 02	
New Tr Term with Diff of Arguments	cos l	cos $\boxed{2D-l}$	cos	cos $\boxed{2D-l}$	cos $\boxed{2D-2l}$	cos l	cos $\boxed{2D-2l}$	cos O	cos l	cos

Co-efficients to be formed	[3] for δg_3				[8] for δk_{302}		[9] for δk_{302}	[10] for δk_{302}	
Factorial Trigonometrical Term	cos O	cos l	cos $\boxed{2D-l}$	cos $\boxed{2D}$	sin f	sin $\boxed{f+l}$	cos f	sin f	sin $\boxed{f-l}$
Variational Trigonometrical Term	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	cos $\boxed{2D-l}$	sin $\boxed{f+l}$	sin $\boxed{f+l}$	cos $\boxed{f+l}$	sin $\boxed{f+l}$	sin $\boxed{f+l}$
Factorial Co-efficient 1 or m, or m^2	− 1 005 837 $-m^2$	+ 163 837 $-m$	+ 028 837 $-m$	+ 021 837 $-m^2$	+ 090	+ 010	− 178 690 3 $+m'$	− 089 489 3 $-m'$	+ 010 489 3 $-m'^2$
		(Figures of multiplication)				(Figures of multiplication)			
PRODUCT	+ 742	− 120	− 021	− 016	+ 090	+ 010	− 355	+ 354	− 040
New Co-efficient	+ 37	− 06	− 01	− 01	− 05	− 01	− 18	− 18	+ 02
New Tr Term with Sum of Arguments	cos $\boxed{2D-l}$	cos $\boxed{2D}$	cos $\boxed{4D-2l}$	cos $\boxed{4D-l}$	cos $\boxed{2f+l}$	cos $\boxed{2f+2l}$	cos $\boxed{2f+l}$	cos $\boxed{2f+l}$	cos $\boxed{2f}$
New Co-efficient	+ 37	− 06	− 01	− 01	+ 05	+ 01	− 18	+ 18	− 02
New Tr Term with Diff of Arguments	cos $\boxed{2D-l}$	cos $\boxed{2D-2l}$	cos O	cos l	cos l	cos O	cos l	cos l	cos $\boxed{2l}$

(The figures in small type are printed in the Form the figures in large type are those in manuscript, peculiar to each application of the Form)

NUMERICAL LUNAR THEORY

SECTION IX

DETAILED FINAL EQUATIONS,

FOR THE FIRST CENTENARY OF ARGUMENTS IN EACH SERIES

PART 1 —DETAILED EQUATIONS DERIVED FROM EQUATION (10)

PART 2 —DETAILED EQUATIONS DERIVED FROM EQUATION (11)

PART 3 —DETAILED EQUATIONS DERIVED FROM EQUATION (12)

NUMERICAL LUNAR THEORY

SECTION IX.—DETAILED FINAL EQUATIONS FOR THE FIRST CENTENARY OF ARGUMENTS IN EACH SERIES

The product-sheets having been completed as is described in Section VIII, the following step was taken.

The results of the product sheets were diligently searched through, attention was given to the first sine or cosine, and for that sine, &c. all the "New Co efficients" in the product sheets were collected, then the operation was repeated for the second sine, &c. forming a separate collection, and this course was continued to the end. Then for each sine, &c. the subordinate co-efficients were examined, the co-efficients of similar quantities were added together, and all these subordinate co-efficients (each consisting of a factor multiplying a variation symbol with subscript attached, taken from the Modified Factorial Table) were collected. Then the subordinate co-efficients were placed in the order of their subscript numbers, to form, for each sine, &c. a grand serial co-efficient. Each serial co efficient was added to the number (which is unattached to any algebraical symbol) derived from summing the corresponding "New Numbers" of the product-sheet, and the sum was made = 0. And thus each linear equation of Section IX as far as No. 100 in each Fundamental Equation was formed.

The discussion of the results here obtained is deferred to Section X.

DETAILED FINAL EQUATIONS, DEDUCED FROM EQUATION (10),

Reference for Argument	Arguments (for Cosines)	Numerical Terms of Equation (10), Column 74	PRODUCTS EXTRACTED FROM
1	0 or nt	− 5 6	
2	l	+ 368	$+ 3\ 99\ \delta g - 0\ 02\ \delta g_1 - 0\ 14\ \delta g_5 - 0\ 02\ \delta g_6$
3	$2D - l$	− 1563 4	$- 0\ 05\ \delta g + 3\ 74\ \delta g_3 - 0\ 11\ \delta g_4 - 0\ 02\ \delta g_{14} - 0\ 02\ \delta g_{19}$
4	$2D$	− 911	$- 0\ 18\ \delta g_3 + 6\ 44\ \delta g_1 - 0\ 39\ \delta g_6 + 0\ 07\ \delta g_{14} - 0\ 05\ \delta g$
5	$2l$	+ 607 3	$- 0\ 22\ \delta g + 6\ 95\ \delta g_5 - 0\ 06\ \delta g_r - 0\ 43\ \delta g_{13} - 0\ 06\ \delta g_0 - 0\ 04\ \delta g_8$
6	$2D + l$	− 647	$- 0\ 51\ \delta g_1 - 0\ 09\ \delta g_5 + 11\ 12\ \delta g_6 - 0\ 82\ \delta g_0 - 0\ 13\ \delta g - 0\ 12\ \delta g_{37}$
7	$2D - S$	− 726	$+ 6\ 17\ \delta g_7 - 0\ 16\ \delta g_8 - 0\ 02\ \delta g_9 - 0\ 35\ \delta g_1 - 0\ 06\ \delta g_{15} - 0\ 05\ \delta g_{19}$
8	$2D - l - S$	+ 71	$- 0\ 10\ \delta g_7 + 3\ 62\ \delta g_8$
9	$l - S$	− 1080 2	$+ 3\ 85\ \delta g_9 - 0\ 02\ \delta g_{15} - 0\ 02\ \delta g_1 - 0\ 13\ \delta g_5$
10	D	+ 627 4	$+ 3\ 86\ \delta g_{10} - 0\ 12\ \delta g_7 - 0\ 02\ \delta g_{10} - 0\ 02\ \delta g_6$
11	$l + S$	+ 450 8	$+ 4\ 14\ \delta g_{11} - 0\ 04\ \delta g_{1r} - 0\ 02\ \delta g_{17} - 0\ 16\ \delta g_9 - 0\ 02\ \delta g_{31}$
12	$2f - l$	− 17	$+ 4\ 04\ \delta g_1 - 0\ 02\ \delta g_{41} - 0\ 02\ \delta g_{18} - 0\ 15\ \delta g_{51}$
13	$3l$	+ 275	$- 0\ 56\ \delta g_5 + 11\ 89\ \delta g_{13} - 0\ 15\ \delta g_0 - 0\ 05\ \delta g_r - 0\ 88\ \delta g_{33} - 0\ 12\ \delta g_{38} - 0\ 09\ \delta g_{53}$
14	$4D - l$	− 2774	$- 0\ 04\ \delta g_3 - 0\ 08\ \delta g_1 + 10\ 38\ \delta g_{14} - 0\ 45\ \delta g_{18} - 0\ 75\ \delta g_{30}$
15	S	− 235 8	$+ 3\ 00\ \delta g_{15}$
16	$2D - l + S$	− 1633 9	$+ 3\ 88\ \delta g_{16} - 0\ 12\ \delta g_{17} - 0\ 02\ \delta g_{51} - 0\ 02\ \delta g_{81}$
17	$2D + S$	− 407	$- 0\ 04\ \delta g_{11} - 0\ 20\ \delta g_{16} + 6\ 72\ \delta g_{17} - 0\ 41\ \delta g_{14} - 0\ 07\ \delta g_{54} - 0\ 06\ \delta g_{80}$
18	$4D - 2l$	− 924	$- 0\ 34\ \delta g_{14} + 5\ 97\ \delta g_{18} - 0\ 15\ \delta g_{6r}$
19	$2D - 2l$	+ 237 7	$- 0\ 03\ \delta g_2 + 3\ 02\ \delta g_{19}$
20	$2D + 2l$	− 550	$- 0\ 10\ \delta g - 0\ 99\ \delta g_6 - 0\ 22\ \delta g_{13} + 17\ 76\ \delta g_0 - 0\ 23\ \delta g_{37} - 1\ 41\ \delta g_{38}$
21	$2D + l - S$	− 940	$- 0\ 48\ \delta g_7 - 0\ 04\ \delta g_9 + 10\ 69\ \delta g_1 - 0\ 08\ \delta g - 0\ 78\ \delta g_{17} - 0\ 12\ \delta g_{19}$
22	$4D$	− 283 3	$- 0\ 10\ \delta g_4 - 0\ 16\ \delta g_6 - 0\ 92\ \delta g_{14} + 16\ 76\ \delta g_2 - 1\ 32\ \delta g_{97}$
23	$D + S$	− 16 7	$+ 4\ 00\ \delta g_3 - 0\ 14\ \delta g_4 - 0\ 02\ \delta g_{73} - 0\ 02\ \delta g_{81}$
24	$2D - 2f$	− 33 4	$+ 3\ 02\ \delta g_1 - 0\ 02\ \delta g_0$
25	$2l - S$	− 352	$- 0\ 20\ \delta g_9 - 0\ 07\ \delta g_1 + 6\ 66\ \delta g_5 - 0\ 06\ \delta g_{17} - 0\ 41\ \delta g_{60} - 0\ 04\ \delta g_{88}$
26	$2D - 3l$	+ 200 8	$- 0\ 03\ \delta g_5 - 0\ 02\ \delta g_{11} - 0\ 04\ \delta g_{19} + 4\ 27\ \delta g_r - 0\ 16\ \delta g_{r1}$
27	$D + l$	+ 231	$- 0\ 20\ \delta g_{10} + 6\ 69\ \delta g_{27} - 0\ 06\ \delta g_{10} - 0\ 04\ \delta g_{r0} - 0\ 41\ \delta g_{63} - 0\ 07\ \delta g_7$
28	$2l - S$	+ 201	$- 0\ 23\ \delta g_{11} + 7\ 26\ \delta g_9 - 0\ 07\ \delta g_{14} - 0\ 02\ \delta g_{1r} - 0\ 46\ \delta g_{61} - 0\ 04\ \delta g_{71} - 0\ 07\ \delta g_{83}$
29	$2D - 2f - l$	+ 17 2	$- 0\ 05\ \delta g_4 + 4\ 33\ \delta g_{20} - 0\ 03\ \delta g_{51} - 0\ 17\ \delta g_r - 0\ 02\ \delta g_{91}$
30	$2D - 2S$	− 247	$+ 5\ 91\ \delta g_{10} - 0\ 15\ \delta g_{12} - 0\ 02\ \delta g_{43} - 0\ 33\ \delta g_{61}$
31	$2D - 2f + l$	− 23 5	$+ 3\ 70\ \delta g_{81} - 0\ 10\ \delta g_6$
32	$2D - l - 2S$	− 28 9	$- 0\ 09\ \delta g_{30} + 3\ 51\ \delta g_3$
33	$4l$	+ 31	$- 1\ 07\ \delta g_{13} + 18\ 81\ \delta g_{33} - 0\ 25\ \delta g_{18} - 0\ 12\ \delta g_{53} - 1\ 50\ \delta g_6$
34	$2D + l + S$	+ 318	$- 0\ 05\ \delta g_{11} - 0\ 54\ \delta g_{14} - 0\ 09\ \delta g_9 + 11\ 55\ \delta g_{34} - 0\ 83\ \delta g_{43} - 0\ 14\ \delta g_8$
35	$4D - l - S$	− 1001	$- 0\ 07\ \delta g_7 - 0\ 04\ \delta g_8 + 9\ 97\ \delta g_{15} - 0\ 43\ \delta g_{30} - 0\ 71\ \delta g_{19}$
36	$3D - l$	+ 281	$- 0\ 02\ \delta g_{10} + 6\ 20\ \delta g_{36} - 0\ 36\ \delta g_{10} - 0\ 16\ \delta g_6$
37	$4D + l$	− 1459	$- 0\ 18\ \delta g_6 - 0\ 27\ \delta g_0 - 1\ 55\ \delta g_{22} + 25\ 12\ \delta g_{37}$
38	$2D + 3l$	− 434	$- 0\ 18\ \delta g_{13} - 1\ 64\ \delta g_0 - 0\ 28\ \delta g_{33} + 26\ 40\ \delta g_{38}$
39	$4D - 2l - S$	− 243	$- 0\ 02\ \delta g_8 - 0\ 32\ \delta g_{15} + 5\ 71\ \delta g_{39}$
40	$3D$	+ 351	$- 0\ 04\ \delta g_{10} - 0\ 08\ \delta g_7 - 0\ 48\ \delta g_{36} + 10\ 74\ \delta g_{40} - 0\ 79\ \delta g_7$
41	$2D + 2f - 2l$	+ 4	$- 0\ 03\ \delta g_{12} + 6\ 53\ \delta g_{11} - 0\ 39\ \delta g_{19}$
42	$D + l + S$	− 66	$- 0\ 22\ \delta g_3 + 6\ 99\ \delta g_1 - 0\ 44\ \delta g_{80} - 0\ 06\ \delta g_{81}$
43	$l - 2S$	− 54 3	$+ 3\ 71\ \delta g_{43} - 0\ 02\ \delta g_{04} - 0\ 10\ \delta g_{90}$
44	$2D - l + 2S$	+ 65 8	$+ 4\ 02\ \delta g_{11} - 0\ 15\ \delta g_9$
45	$2D - 2l - S$	+ 61 0	$+ 3\ 04\ \delta g_{45} - 0\ 02\ \delta g_{71}$
46	$2S$	− 35 5	$+ 3\ 02\ \delta g_{46} - 0\ 02\ \delta g_{r0}$
47	$2D + 2l - S$	− 425	$- 0\ 95\ \delta g_1 - 0\ 10\ \delta g_5 + 17\ 20\ \delta g_{47} - 0\ 17\ \delta g_{60}$
48	$2D + 2f - l$	+ 3	$- 0\ 05\ \delta g_1 - 0\ 52\ \delta g_{11} + 11\ 26\ \delta g_{18} - 0\ 09\ \delta g_{51}$
49	$4D - S$	− 909	$- 0\ 09\ \delta g_7 - 0\ 15\ \delta g_{21} - 0\ 88\ \delta g_{35} + 16\ 21\ \delta g_{49}$
50	$l + 2S$	+ 5 1	$- 0\ 05\ \delta g_{46} + 4\ 31\ \delta g_{60} - 0\ 17\ \delta g_{91} - 0\ 03\ \delta g_{92}$

SECTION IX, PART 1 —DETAILED FINAL EQUATIONS

EACH LINE SEPARATELY = 0

THE PRODUCT SHEETS		Reference for Argument
		1
	+ 0 05 δk_{309} + 0 05 δk_{103}	2
− 1 71 δh_{3}	+ 0 04 δk_{305} − 0 04 δk_{106}	3
− 3 68 δh_{4}	− 0 16 δk_{304} − 0 02 δk_{305} + 0 16 δl_{307} + 0 02 δk_{301}	4
− 3 94 δh_{5}	− 0 02 δk_{30} + 0 18 δk_{309} + 0 18 δk_{310} − 0 02 δk_{319}	5
− 5 65 δh_{6}	− 0 04 δk_{307} − 0 36 δk_{303} + 0 37 δl_{313} + 0 04 δk_{314}	6
− 3 53 δh_{7}	− 0 15 δk_{311} − 0 02 δk_{31} + 0 14 δk_{310} + 0 02 δl_{316}	7
− 1 56 δh_{8}	+ 0 03 δk_{315} − 0 03 δl_{317}	8
− 1 83 δh_{9}	+ 0 04 δk_{319} + 0 04 δk_{3}	9
− 1 83 δh_{10}	+ 0 05 δk_{3} − 0 04 δk_{3}	10
− 2 12 δh_{11}	+ 0 06 δk_{324} + 0 06 δk_{325}	11
− 2 02 δh_{12}	− 0 05 δk_{303} + 0 05 δk_{331}	12
− 5 92 δh_{13} + 0 02 δh_{6}	− 0 04 δk_{309} + 0 40 δk_{318} + 0 40 δk_{319}	13
− 5 39 δh_{14}	− 0 33 δk_{318} + 0 04 δk_{3} + 0 33 δk_{330} − 0 04 δk_{33}	14
− 0 15 δh_{15}		15
− 1 86 δh_{16}	+ 0 05 δk_{335} − 0 04 δk_{31}	16
− 3 83 δh_{17}	− 0 17 δk_{314} − 0 02 δk_{345} + 0 17 δk_{311} + 0 02 δk_{347}	17
− 3 42 δh_{18}	+ 0 14 δk_{38} − 0 13 δk_{355}	18
+ 0 26 δh_{19}		19
+ 0 02 δh_{5} − 7 62 δh_{20}	− 0 07 δk_{313} − 0 66 δk_{331} + 0 66 δk_{330} + 0 07 δk_{39}	20
− 5 50 δh_{21}	− 0 04 δk_{316} − 0 35 δk_{340} + 0 34 δk_{344} + 0 04 δk_{383}	21
+ 0 02 δh_{4} − 7 36 δh_{22}	− 0 61 δk_{307} − 0 07 δk_{390} + 0 61 δk_{346} + 0 07 δk_{357}	22
− 1 99 δh_{23}	− 0 05 δk_{348} + 0 05 δk_{311}	23
+ 0 31 δh_{24}		24
− 3 80 δh_{25}	− 0 02 δk_{319} + 0 17 δk_{350} + 0 17 δk_{365} − 0 02 δk_{400}	25
+ 2 24 δh_{26}	− 0 06 δk_{310} + 0 06 δk_{31}	26
− 3 81 δh_{27}	− 0 02 δk_{319} + 0 17 δk_{39} − 0 17 δk_{361}	27
− 4 09 δh_{28}	− 0 02 δk_{321} + 0 20 δk_{353} + 0 19 δk_{363} − 0 02 δl_{300}	28
+ 2 29 δh_{29}	+ 0 06 δk_{406} − 0 07 δk_{30}	29
− 3 38 δh_{30}	− 0 13 δk_{343} + 0 14 δk_{360}	30
− 1 66 δh_{31}	+ 0 04 δk_{309} − 0 03 δk_{360}	31
− 1 41 δh_{32}	+ 0 03 δk_{361} − 0 02 δk_{369}	32
− 7 89 δh_{33} + 0 02 δh_{53}	− 0 08 δk_{318} + 0 71 δk_{308}	33
− 5 80 δh_{34}	− 0 04 δk_{311} − 0 38 δk_{317} + 0 38 δk_{371}	34
− 5 24 δh_{35}	− 0 32 δk_{356} + 0 03 δk_{358} + 0 32 δk_{370} − 0 04 δk_{375}	35
− 3 55 δh_{36}	+ 0 02 δk_{361} − 0 15 δk_{307} + 0 15 δk_{318}	36
− 0 02 δh_{6} − 9 33 δh_{37}	− 0 11 δk_{345} − 0 99 δk_{357} + 0 99 δk_{317}	37
− 0 02 δh_{13} − 9 60 δh_{38}	− 0 12 δk_{349} − 1 04 δk_{378} + 1 04 δk_{38}	38
− 3 27 δh_{39}	+ 0 12 δk_{389}	39
− 5 52 δh_{40}	− 0 35 δk_{361} − 0 04 δk_{370}	40
− 3 73 δh_{41}	+ 0 02 δk_{305} − 0 16 δk_{337}	41
− 3 96 δh_{42}	− 0 02 δk_{349} + 0 18 δk_{380}	42
− 1 68 δh_{43}	+ 0 04 δk_{393} + 0 04 δk_{309}	43
− 2 00 δh_{44}		44
+ 0 41 δh_{45}		45
− 0 30 δh_{46}		46
− 0 02 δh_{6} − 7 48 δh_{47}	− 0 64 δk_{383} + 0 64 δk_{390}	47
− 5 70 δh_{48}	− 0 38 δk_{306} + 0 04 δk_{307} + 0 37 δk_{31}	48
− 0 02 δh_{7} − 7 12 δh_{49}	− 0 59 δk_{358} − 0 07 δk_{370} + 0 59 δk_{388}	49
− 2 27 δh_{50}	+ 0 06 δk_{304} + 0 06 δk_{397}	50

DETAILED FINAL EQUATIONS, DEDUCED FROM EQUATION (10),

Column for Argument	Arguments (for Cosines)	Numerical Terms of Equation (10), Column 74	PRODUCTS EXTRACTED FROM
51	$2f$	+ 61 5	$-0\ 22\ \delta g_{1} - 0\ 04\ \delta g_{20} - 0\ 07\ \delta g_{18} + 7\ 05\ \delta g_{51} - 0\ 44\ \delta g_{91}$
52	$D - 2f - 2l$	+ 24	$-0\ 26\ \delta g_{9} + 7\ 61\ \delta g_{5} - 0\ 08\ \delta g_{33}$
53	$2D - 4l$	+ 73	$-0\ 07\ \delta g_{13} - 0\ 25\ \delta g_{6} - 0\ 07\ \delta g_{33} + 7\ 50\ \delta g_{53}$
54	$4D - l + S$	+ 171	$-0\ 04\ \delta g_{16} - 0\ 08\ \delta g_{17} + 10\ 79\ \delta g_{54} - 0\ 48\ \delta g_{58} - 0\ 79\ \delta g_{85}$
55	$D - l$	− 38 6	$+3\ 00\ \delta g_{5}$
56	$D - 2f$	+ 1 7	$+4\ 18\ \delta g_{56}$
57	$2D - 2f - S$	+ 8 8	$+3\ 05\ \delta g_{57} - 0\ 02\ \delta g_{63}$
58	$4D - 2l + S$	+ 25	$-0\ 02\ \delta g_{16} - 0\ 36\ \delta g_{1} + 6\ 23\ \delta g_{58}$
59	$D - 2l$	− 76 1	$-0\ 02\ \delta g_{7} - 0\ 04\ \delta g_{8} + 4\ 12\ \delta g_{59} - 0\ 02\ \delta g_{63}$
60	$3l - S$	− 131	$-0\ 53\ \delta g - 0\ 14\ \delta g_{17} + 11\ 45\ \delta g_{60} - 0\ 05\ \delta g_{88}$
61	$3l + S$	+ 68	$-0\ 60\ \delta g_{8} + 12\ 34\ \delta g_{61} - 0\ 06\ \delta g_{71} - 0\ 15\ \delta g_{83}$
62	$D - f + 2l$	− 15	$+0\ 02\ \delta g_{0} - 0\ 18\ \delta g_{31} + 6\ 35\ \delta g_{62} - 0\ 03\ \delta g_{97}$
63	$D + 2l$	+ 92	$-0\ 54\ \delta g_{7} - 0\ 05\ \delta g_{59} + 11\ 50\ \delta g_{63} - 0\ 14\ \delta g_{77}$
64	$2D + l - 2S$	− 153	$-0\ 45\ \delta g_{30} - 0\ 04\ \delta g_{43} + 10\ 28\ \delta g_{64} - 0\ 08\ \delta g_{90}$
65	$3D - 2l$	+ 25 6	$-0\ 10\ \delta g_{36} + 3\ 63\ \delta g_{65}$
66	$4D - 3l$	+ 0 4	$-0\ 09\ \delta g_{18} + 3\ 53\ \delta g_{66}$
67	$4D - 2f$	+ 9	$-0\ 02\ \delta g_{31} + 5\ 88\ \delta g_{67} - 0\ 15\ \delta g_{70}$
68	$5l$	− 79	$-1\ 74\ \delta g_{39} + 27\ 70\ \delta g_{68}$
69	$2D - 2f - l - S$	+ 2 3	$-0\ 05\ \delta g_{67} + 4\ 50\ \delta g_{69} - 0\ 03\ \delta g_{77}$
70	$4D - 2f - l$	+ 7 1	$-0\ 08\ \delta g_{6} + 3\ 49\ \delta g_{70}$
71	$2D - 3l - S$	+ 22 3	$-0\ 03\ \delta g_{9} - 0\ 05\ \delta g_{45} - 0\ 03\ \delta g_{11} + 4\ 44\ \delta g_{71}$
72	$3D + l$	+ 55	$-0\ 10\ \delta g_{7} - 0\ 96\ \delta g_{10} - 0\ 17\ \delta g_{63} + 17\ 26\ \delta g_{72}$
73	$3D - l + S$	− 12	$-0\ 03\ \delta g_{3} + 6\ 47\ \delta g_{73} - 0\ 38\ \delta g_{81}$
74	$2D - 2f + l - S$	+ 1 9	$+3\ 58\ \delta g_{74}$
75	$D - S$	+ 16 0	$+3\ 73\ \delta g - 0\ 02\ \delta g_{97} - 0\ 11\ \delta g_{30}$
76	$2f - S$	+ 13	$+6\ 76\ \delta g_{76} - 0\ {-0}\ \delta g_{78} - 0\ 04\ \delta g_{89}$
77	$f + S$	+ 35	$-0\ 02\ \delta g_{87} - 0\ 04\ \delta g_{69} + 7\ 36\ \delta g_{77} - 0\ 25\ \delta g_{79}$
78	$2f - l - S$	− 6 0	$-0\ 12\ \delta g_{76} + 3\ 89\ \delta g_{78}$
79	$2f - l + S$	+ 9 4	$-0\ 16\ \delta g_{77} + 4\ 20\ \delta g_{79}$
80	$D + 2l + S$	+ 16	$-0\ 56\ \delta g_{4} + 11\ 94\ \delta g_{80}$
81	$3D + S$	− 16	$-0\ 05\ \delta g_{23} - 0\ 09\ \delta g_{1} - 0\ 51\ \delta g_{73} + 11\ 16\ \delta g_{81}$
82	$3D - 2f$	− 4 7	$+3\ 59\ \delta g_{8}$
83	$2D + 2l + S$	+ 159	$-0\ 10\ \delta g_{8} - 1\ 03\ \delta g_{81} - 0\ 18\ \delta g_{61} + 18\ 35\ \delta g_{83}$
84	$2D - 2l + S$	+ 23 3	$+3\ 00\ \delta g_{84}$
85	$4D + S$	+ 193	$-0\ 10\ \delta g_{17} - 0\ 17\ \delta g_{34} - 0\ 96\ \delta g_{64} + 17\ 33\ \delta g_{85}$
86	$D - 2l + S$	+ 9 8	$+3\ 97\ \delta g_{8} - 0\ 02\ \delta g_{96}$
87	$3D - S$	− 1	$-0\ 04\ \delta g_{75} + 10\ 33\ \delta g_{87} - 0\ 08\ \delta g_{96}$
88	$2D - 3l + S$	+ 23 1	$-0\ 0\text{-}\ \delta g_{25} - 0\ 02\ \delta g_{60} - 0\ 04\ \delta g_{81} + 4\ 11\ \delta g_{88}$
89	$2D - 2f - l + S$	+ 4 5	$-0\ 02\ \delta g_{76} + 4\ 16\ \delta g_{89} - 0\ 04\ \delta g_{11}$
90	$2l - 2S$	− 23	$-0\ 18\ \delta g_{43} - 0\ 06\ \delta g_{64} + 6\ 38\ \delta g_{90}$
91	$2l + 2S$	− 1	$-0\ 26\ \delta g_{50} + 7\ 57\ \delta g_{91}$
92	$2D + 2S$	+ 20 8	$-0\ 22\ \delta g_{44} - 0\ 04\ \delta g_{50} + 7\ 02\ \delta g_{9}$
93	$f + l$	− 2	$-0\ 05\ \delta g_{29} - 0\ 57\ \delta g_{61} - 0\ 09\ \delta g_{52} + 12\ 04\ \delta g_{93}$
94	$2D - 2f + S$	− 2 0	$+3\ 00\ \delta g_{94}$
95	$2D - 2f + l + S$	+ 0 4	$-0\ 02\ \delta g_{91} + 3\ 83\ \delta g_{95}$
96	$D + l - S$	+ 38	$-0\ 18\ \delta g_{75} - 0\ 03\ \delta g_{86} - 0\ 06\ \delta g_{87} + 6\ 41\ \delta g_{96}$
97	$2f - 3l$	− 2 3	$-0\ 02\ \delta g_{62} + 3\ 94\ \delta g_{97}$
98	$D - 2S$	+ 3 2	$+3\ 60\ \delta g_{98}$
99	$D - 2f + S$	− 2	$+4\ 02\ \delta g_{99}$
100	$D + 2S$	− 0 7	$+4\ 16\ \delta g_{100}$

SECTION IX, PART 1 —DETAILED FINAL EQUATIONS

EACH LINE SEPARATELY = 0

THE PRODUCT SHEETS		Reference for Argument
− 3 99 δh_{51}	− 0 18 δh_{301} + 0 02 δk_{30} + 0 18 δk_{31} − 0 02 δk_{331}	51
+ 4 26 δh_{5}	+ 0 21 δk_{31} − 0 02 δk_{340} + 0 02 δk_{36}	52
+ 4 21 δh_{53}	+ 0 02 δk_{310} + 0 02 δk_{37} − 0 20 δk_{386}	53
− 5 54 δh_{54}	− 0 35 δk_{374} + 0 04 δk_{387}	54
+ 0 13 δh_{5}		55
+ 2 15 δh_{56}	+ 0 06 $o k_{37}$	56
+ 0 46 δh_{57}		57
− 3 56 $o h_{59}$	+ 0 02 δk_{374}	58
[illegible] 10 δh_{59}		59
− 5 77 δh_{60}	− 0 04 δk_{350} + 0 38 δk_{395} + 0 38 δk_{400}	60
− 6 06 δh_{61} + 0 02 δh_{71}	− 0 05 δk_{353} + 0 42 δl_{306} + 0 42 δk_{399}	61
− 3 63 δh_{6}	− 0 02 δl_{309} + 0 16 δk_{331}	62
− 5 78 δh_{63}	− 0 04 δk_{85}	63
− 5 35 δh_{64}	− 0 04 δk_{300}	64
− 1 58 δh_{65}		65
− 1 44 δh_{66}		66
− 3 37 δh_{67}	+ 0 13 δk_{10}	67
− 9 86 δh_{68}	− 0 12 δh_{108}	68
+ 2 44 δh_{69}	+ 0 07 δk_{11}	69
− 1 39 δh_{70}	+ 0 03 δh_{119}	70
+ 2 38 δh_{71}		71
− 0 02 δh_{7} − 7 49 δh_{72}		72
− 3 70 δh_{73}		73
− 1 51 δh_{74}	+ 0 03 δk_{310}	74
− 1 69 δh_{75}		75
− 3 84 δh_{76}	+ 0 02 δl_{319} − 0 17 δk_{86}	76
− 4 14 δh_{77}	− 0 20 δk_{80} + 0 02 δk_{324}	77
− 1 87 δh_{78}	− 0 04 δk_{35}	78
− 2 17 δh_{79}	− 0 06 δk_{10}	79
− 5 93 δh_{80}	− 0 04 δk_{180}	80
− 5 67 δh_{81}		81
− 1 53 δh_{82}	+ 0 03 δh_{181}	82
− 0 02 δh_{29} − 7 77 δh_{83}	− 0 08 δk_{313}	83
+ 0 12 δh_{84}		84
− 0 02 δh_{17} − 7 51 δh_{85}	− 0 64 δk_{387}	85
+ 1 96 δh_{86}		86
− 5 37 δh_{87}		87
+ 2 09 δh_{88}		88
+ 2 14 δh_{89}	+ 0 06 δk_{11}	89
− 3 65 δh_{90}	− 0 02 δk_{391}	90
− 4 24 δh_{91}	− 0 02 δk_{394}	91
− 3 98 δh_{9}	− 0 18 δk_{381}	92
+ 0 02 δh_{20} − 5 97 δh_{93}	− 0 41 δk_{30} + 0 04 δk_{308} − 0 05 δk_{31} + 0 40 δk_{346}	93
+ 0 17 δh_{94}		94
− 1 81 δh_{95}	+ 0 04 δk_{347}	95
− 3 66 δh_{96}		96
+ 1 92 δh_{97}	− 0 05 δk_{338}	97
− 1 54 δh_{98}		98
+ 2 00 δh_{99}	+ 0 05 δk_{319}	99
− 2 14 δh_{100}		100

DETAILED FINAL EQUATIONS, DERIVED FROM EQUATION (

Reference No. of Argument	Arguments (for Sine)	Numerical Terms of Equation ( ) Column 75	PRODUCTS EXTRACTED F
3, 4, 5	D l; D l; l	92 7; 725 9; 3 3; 345	[illegible]
6, 7, 8, 9	D l; D l g; D l g; D g	385; 409; 39; 508 4; 198 5	[illegible]
3, 4, 5	l g; f l; D $3l$; l g	+ 34 9; 41; 1496; 9 4	[illegible]
6, 7, 8, 9, 10	D l g; D + g; $4D$ l; D l; D l	783 5; 20; 55; 6 4; 249	[illegible]
13, 14, 15	D l g; $4D$; D g; D f l g	5 4; 324; 8; 3; 98	[illegible]
26, 27, 28, 29, 30	D $3l$; D l; l g; D f l; D g	3 3; 7; 3; 38	[illegible]
31, 32, 33, 34, 35	D f l; D l g; $4l$; D l g; $4D$ l g	9 6; 5; 7; 70; 502	[illegible]
36, 37, 38, 39, 40	$3D$ l; $4D$ l; D $3l$; $4D$ l g; $3D$	− 5; 5; 55; 48; 49	[illegible]
41, 42, 43, 44, 45	D f l; D l g; l g; D l g; D $-4l$ g	3; − 34; 24 5; 23 3; − 3 4	[illegible]
46, 47, 48, 49, 50	D g; D $4l$ g; $4D$; l g	3; 72; 3; 342; 3	[illegible]

DACH LINI SLJ ARAIELY =

THD IRODUCT SHDD18

													Ref
− 3 93	3 43 6	9 3	5 7	6		9 8 7 8	−	8 6 8	−	h			3 4 5
− 84	6 3	8 8 6	59 5	6		5 6 7 8	− −	5 6 9					6 7 8 9
− 8 86 4	7 35	3 4 5	64		6	9 9 6 4		9 7 4					3 4 5
5	3 7 93	5 7	6			− 8 7 5	− −	8 5					6 7 8
59		4 7	7				−		34	34			9
3	66 55	5 3 7 3	57 4	9		− 5 9	−	5 33 9		33			3
5	3 65	6	3			7		7					4 5
− 3 68 4 4 7	6 5 6 3	3 34 4 4	3	6 7		− 7 5	− −	7 8 5					6 7 8 9
− 5 75 3 − 6 94	8 6 4	7 8 5 5	6 6			7 6 6 4	− −	6 6 3	35				3 3 33 34 35
− 3 9 7 4 5	7 94 8 7 7	7 3 3 4 7 7	57			6 3 5	− −	6	− − 4 43	4 41			36 37 38 39 4
− 5 − 3 97 − 7	3 3	6 8				7 8 7		8					4 4 43 44 45
7 9 − 9 − 3	4 4 8 5	6 3 6				6	− −	5 3	34	33 3			46 47 48 49 5

DETAILED FINAL EQUATIONS DERIVED FROM EQUATION ( ):

Reference of Argument	Arguments (for Moon)				Numerical Terms of Equation ( ) Column 75	PRODUCTS EXTRACTED FROM
51		f			35 6	[illegible]
52	D	f	l		7	[illegible]
53	D		$4l$		40	[illegible]
54	$4D$		l	g	26	[illegible]
55	D		l		8	[illegible]
56	D	f			5	[illegible]
57	D	f		g	5	[illegible]
58	$4D$		l	g	1	[illegible]
59	D		l		38 5	[illegible]
60			$3l$	g	65	[illegible]
61			$3l$	g	7	[illegible]
62	D	f	l		7	[illegible]
63	D		l		43	[illegible]
64	D		l	g		[illegible]
65	$3D$		l			[illegible]
66	$4D$		$3l$		6	[illegible]
67	$4D$	f			8	[illegible]
68			$5l$		40	[illegible]
69	D	f	l	g	6	[illegible]
70	$4D$	f	l		3 7	[illegible]
71	D		$3l$	g	8	[illegible]
72	$2D$		l		5	[illegible]
73	$2D$		l	g	9	[illegible]
74	D	f	l −	g		[illegible]
75	D			g	7	[illegible]
76		f		g	9	[illegible]
77		f		g		[illegible]
78		f	l −	g	1	[illegible]
79		f	l	g	5	[illegible]
80	D		l	g	9	[illegible]
81	$3D$			g		[illegible]
82	$3D$	f				[illegible]
83	D		l	g	57	[illegible]
84	D		l	g	4	[illegible]
85	$4D$			g	80	[illegible]
86	D		l	g	5	[illegible]
87	$3D$			g	9	[illegible]
88	D		$3l$	g	9	[illegible]
89	D	f	l	g		[illegible]
90			l	g	3	[illegible]
91			l	g		[illegible]
92	D			g	8	[illegible]
93		f	l			[illegible]
94	D	f		g		[illegible]
95	D	f	l	g	3	[illegible]
96	D		l	g	6	[illegible]
97		f	$3l$			[illegible]
98	D			g	5	[illegible]
99	D	f		g		[illegible]
100	D			g	3	[illegible]

Sdotion IX Part 2 —dietailed ityal equations

DACII LINE SEPARATIIY =

PHI IRODUOT 5H I I5

									R f Arum
/	8/	6 8/	4 8/	33 /	8/	8 8/			5
3 8/	1 59 /	6 /			9 /				5
/	3 8/	/	4 4 /		9 /				53
3 /	6 /	7 76 8/	7 /	57 8/	3 8/				54
									53
8/					8/				56
5 8/					8/				57
8/	7 8/	8/							58
/	3 /								
3 8/	8/	8 4 /	3 8/		6 /	6 8/			6
34 8/	9 3 8/	4 /	8/		8/	/	7 8/	7 8/	6
8/	3 3 8/	8/			6 /				6
3 8/	3 8/	8 46 8/	8/						3
5 8/	8/	7 5 8/	4 /						64
7 8/	63 /								65
7 8/	53 /								66
8/	/	87 8/	7 /		5 8/				67
6 8/	4 6 8/				8/				68
5 /	8/				8/				69
6 /	49 8/				6 8/				7
8/	4 8/	44 /							
7 8/	57 8/	8/	4 8/		8/				7
/	3 46 8/	9 8/							73
58 /					7 8/				7
7 8/	8/	8 8/							75
3 74 8/	8/	8/			7 8/				76
8/	4 34 8/	8/			9 8/				77
8/	89 8/				8 /				8
/	9 8/				/				79
3 8/	– 8 9 /								8
3 8/	6 8/	9 8/	8 3 8/						8
59 8/					8/				8
7 8/	6 8/	/	5 9 8/		8/	8/			83
7 8/	8/	5 8/	4 7 8/		/	34 8/ 89			84 85
97 8/	8/								86
3 /	7 3 8/	/							8
3 8/	8/								88
8/	5 8/				8/				89
9 8/	5 8/	3 36 8/							9
3 /	4 56 8/								9
8/	8/	4 8/			8 8/				9
4 8/	33 /	6 8/	9 8/		8/	– 7 8k	8h	7 8/	93
85 8/					8 8h				94 95
9 8/	/	5 8/	– 3 4 8/						96
8/	94 /				9 /				97
6 /									98
8/					9 8/				99
6 /									

DETAILED FINAL EQUATIONS, DEDUCED FROM EQUATION ( ):

Reference for Argument	Arguments (for Sines).				Numerical Terms of Equation ( )	PRODUCTS EXTRACTED FROM			
30		*f*			5	3 [illegible]			
302		*f*	*l*		+	26 [illegible]	02 [illegible]	02 [illegible]	6 [illegible]
3 3		*f*	*l*		5	08 [illegible] −	09 [illegible]		
304	*D*	*f*			8	[illegible]	[illegible]		
305	*D*	*f*	*l*		69	24 [illegible]	02 [illegible]	02 [illegible]	23 [illegible]
306	*D*	*f*	*l*		43	08 [illegible]	9 [illegible]		
307	*D*	*f*			60	44 [illegible]	04 [illegible]	04 [illegible]	
308		*f*	*l*		5	48 [illegible] −	04 [illegible]	5 [illegible]	
309	*D*	*f*	*l*			84 [illegible]	3 [illegible]	02 [illegible]	
3		*f*	*l*		8	14 [illegible]			
	D	*f*		*g*	3	[illegible]	[illegible]		
3	*D*	*f*	*l*		9 8	14 [illegible]	4 [illegible]		
3 3	*D*	*f*	*l*		64	74 [illegible] −	07 [illegible]		
3 4	*D*	*f*		*g*	4	[illegible]	[illegible]		
3 5	*D*	*f*	*l*	*g*	4	02 [illegible]	3 [illegible]		
3 6	*D*	*f*		*g*	5	43 [illegible]	04 [illegible]		
3 7	*D*	*f*	*l* −	*g*		09 [illegible]	09 [illegible]		
3 8	4*D*	*f*	*l*		7	− [illegible]	[illegible]		
3 9		*f*	*l*	*g*	38	5 [illegible] −	02 [illegible]	02 [illegible]	
320		*f*		*g*	4 7	4 [illegible]	3 [illegible]		
3		3*f*			4	49 [illegible]	5 [illegible]		
3		*f*	*l*	*g*	4	7 [illegible]	8 [illegible]		
323	*D*	*f*				3 [illegible]	02 [illegible]		
324		*f*	*l*	*g*	7	28 [illegible] −	02 [illegible]	02 [illegible]	
325		*f*	*l* −	*g*	7	08 [illegible] −	09 [illegible]		
326		*f*	−	*g*	− 7 4	+ [illegible] −	3 [illegible]		
3 7	*D*	*f*			7	− 09 [illegible]	08 [illegible]		
3		*f* + 3*l*				78 [illegible] −	08 [illegible]		
329	4*D*	*f*			87	04 [illegible]	4 [illegible]	46 [illegible]	
330	4*D*	*f*	*l*		− 04	70 [illegible] −	07 [illegible]		
33		3*f*	*l*		8	7 [illegible]	06 [illegible]		
332	4*D*	*f*	*l*		26	04 [illegible] +	43 [illegible]		
333	*D*	3*f*			4	4 [illegible]			
334	*D*	*f*	*l*			04 [illegible]	44 [illegible]	44 [illegible]	
335	*D*	*f*	*l*	*g*	55	3 [illegible]	02 [illegible]		
336	*D*	*f*	*l*	*g*		02 [illegible] −	[illegible]	3 [illegible]	
337	*D*	*f*	*l*		8	[illegible] −	[illegible]		
338		*f*	3*l*		87	− 06 [illegible]	26 [illegible]	02 [illegible] −	26 [illegible]
339	*D*	*f*	*l*		24	[illegible]	[illegible]		
340	*D*	*f*	2*l*			29 [illegible]	02 [illegible]	02 [illegible]	
[illegible]	*D*	*f*		*g*		46 [illegible] −	04 [illegible]		
[illegible]	*D*	*f*	*l*	*g*	20	− 09 [illegible]	08 [illegible]		
[illegible]	*D*	*f*	−	*g*	5	[illegible]			
[illegible]	[illegible]	*f*	*l*	*g*	3	7 [illegible] −	67 [illegible]		
[illegible]	[illegible]	*f*			77	67 [illegible]	63 [illegible] −	[illegible]	
[illegible]	[illegible]	3*f*	*l*		4	80 [illegible]			
[illegible]	[illegible]	*f*	*l*	*g*	8	06 [illegible]	3 [illegible]	23 [illegible]	
[illegible]	*D*	*f*		*g*		08 [illegible] +	09 [illegible]		
[illegible]	[illegible]	*f*		*g*		6 [illegible]	02 [illegible]		
[illegible]		*f* +	*l* −	*g*	7	46 [illegible] −	04 [illegible]		

EACH LINE SEPARATELY =

THE PRODUCT SHALL BE

							Ref Arm
3 8/ 8/ / 9 8/ 3 /	8 8/ 9 8/ 8/ 7 / /	8/ 6 / 8/ 8/	3 8/ 5 8/	6 / 3 /	3 8/ 3 /	8/ 5 8	3 3 4 5
5 / 3 8/ 5 / 5 / 8 8/	99 / 5 8/ 9 / 37 / 8/	6 8/ 7 8/ 5 8/ 8/ 5 8/	7 4 8/ 5 8/ 3 / 8/	7 8/ 3 / 5 /	5 / 8 88	5 8/ 5 /	3 6 3 3 3 9 3
4 8/ 9 8/ 8/ 8/ — 9 8/	7 / 8/ 45 8/ 8 8/ 4 8/	6 8/ 8 8/ 7 8/ 8 / 8/	9 8/ 3 7 8/ 8/ 8	45 8/ 8	8 8/ 3 88	7 /	3 3 3 3 4 5
7 8/ 6 8/ 8/ 3 8/ — 5 8/	6 7 8/ 96 8/ 8/ 68 8/ 8 8/	5 8/ 5 8/ 9 8/ 5 / 9 8/	3 8/ 3 8/ /	6 88 3 88 3 88	5 8/ 5 88	5 88	3 6 3 7 3 8 3 9 3
— 8 6 8/ 5 8/ — 7 8/ — 8/ — 8/	3 8/ 8/ — 8/ 7 8/ 5 8/	8 8/ 5 / 5 8/ 3 7 8/ 5 /	9 8/ 3 8/ 3 /	5 / 3 8/ 3 8/	5 8/ 5 8/ 7 8/	4 88 3 88	3 3 3 3 3 4 3 5
8/ 8/ 48 8 5 8/ 6 8/	8 8/ 5 8/ 4 8/ — 6 6 / 7 8/	8 88 6 / 9 8/ 5 / 78 8/	5 8/ 7 88 43 /	47 / 43 8/	8 88	8 88	3 3 7 3 8 3 9 33
6 8/ 5 8/ — / 7 8/ 3 8/	3 7 8/ 5 8/ 31 8/ 5 8/ 8/	4 8/ — 6 4 8/ 9 8/ — 99 8/ 75 8/	8/ 3 / 9 8/ 5 8/ 5 /	3 / 5 8/	7 88 6 88		33 33 333 33 335
4 8/ 7 / 6 8/ / 8/	8/ 5 8/ 3 8/ 69 8/ 8 88	8/ 7 8/ 87 8/ 8/ 3 8/	8/ 3 / 1 / 3 5 8/	/ 8 7 8/	3 8/ 6 88		33 337 338 339 34
3 8/ 5 8/ 5 / 44 8/ 9 8/	5 8/ 8/ 6 8/ 6 / 8/	8 8/ 3 8/ 3 8/	7 5 88 7 8/ 65 /	9 88 8 88	43 /		3 4 43 344 345
4 8/ 5 8/ / — 3 8/ 8 88	5 8/ 8/ 6 8/ 5 88	— / 65 8/ — 7 47 8/	9 / 3 88 5 8/	48 8/ 3 88 8 /			34 34 3 8 49 35

DETAILED FINAL EQUATIONS, DEDUCED FROM EQUATION ( ):

Reference Argument	Arguments (for Sun)				Numerical Terms of Equation ( )	PRODUCTS EXTRACTED FROM		
35[illegible]	D	f	l	S	5	5 [illegible]		
35[illegible]	D	f	l		3	46 [illegible]	04 [illegible]	
353		f	l	S	4	5 [illegible]	5 [illegible]	
354	D	f	l		5 3	[illegible]		
355	$4D$	f	l		00	[illegible]		
356	$4D$	f	l	S	24	21 [illegible]		
357	D	f	l		5	07 [illegible]		
358	$4D$	f		S	26	39 [illegible]		
359	D	f	l		7	[illegible]		
360	D	f		S		4 [illegible]	04 [illegible]	
36[illegible]	$3D$	f			5	00 [illegible]	22 [illegible]	3 [illegible]
36[illegible]	D	$3f$	l		5	29 [illegible]	00 [illegible]	
363		f	l −	S	3 7	4 [illegible]		
364	D	f	l	S	5	[illegible]		
365		f	l	S	5 7	[illegible]		
366	D	$3f$	l			09 [illegible]		
367	$3D$	f	l		4	[illegible]		
368		f	$4l$		9	8 [illegible]		
369	D	f	l	S		09 [illegible]		
370	$4D$	f	l	S	26	67 [illegible] −	07 [illegible]	
3[illegible]	D	$3f$	l			75 [illegible]		
3[illegible]	D	f	$3l$		4	09 [illegible]		
373	D	f	l	S	22	77 [illegible]	8 [illegible]	
374	$4D$	f	l	S		[illegible]	00 [illegible]	
375	$4D$	f	l	S	6	03 [illegible]	40 [illegible]	
376	$3D$	f	l			43 [illegible]		
377	$4D$	f	l		29	35 [illegible]		
378	D	f	$3l$		3	07 [illegible]	74 [illegible]	
379	D	$3f$						
380	D	f	l		3	[illegible]		
38[illegible]		$3f$	l					
38[illegible]	D	f	$3l$		24 4	+ 60 [illegible]		
383	D	f	l	S		04 [illegible]	43 [illegible]	
384	D	f		S		3 [illegible]		
385		$3f$	l					
386	D	f	$4l$		3	5 [illegible]		
387	$4D$ −	f		S		+ 04 [illegible]	44 [illegible]	
388	$4D$	f		S	65	3 [illegible]		
389	D	f	l	S	3	49 [illegible]	04 [illegible]	
390	D	f	l	S	24	+ 09 [illegible]		
[illegible]		f		S		+ [illegible]		
[illegible]		f		S		[illegible]		
[illegible]		f	l	S		4 [illegible]		
[illegible]		f	l	S	3	3 [illegible], 29 [illegible]	00 [illegible]	
[illegible]		f	$3l$	S	8	78 [illegible]		
[illegible]		f	$3l$	S		8 [illegible]		
[illegible]		f	l	S		29 [illegible]		
[illegible]		f	l	S		09 [illegible]		
[illegible]		f	$3l$	S		00 [illegible]	7 [illegible]	
[illegible]		f	$3l$	S	6	00 [illegible]		

EACH LINE SEPARATELY =

THE PRODUCT SHEETS

					Reference
9 8/	46 8/	— 8/			
8 8/	— 7 5 8/	3 8/			35
3 8/	5 /	8 37 /	6 8/	3 8/	35
8 8/	8 /	/			353
— 3 /	7 8/	5 8/			354 355
/	/	65 8/	3 8/		
6 8/	43 8/	7 /	6 /		356
3 8/	8/	/	4 8/	5 86 8/	35
8 8/	3 8/				358
— 6 3 8/	5 8/	3 8/	4 /		359 56
8/	8/	3 8/	1 8/		
— 1 8/	8 8/	3 8/	3 63 8/		36
8 8/	8/	8 8/			36
3 8/	9 8/	8/	8/		363
8 8/	8/	9 8/	8 8/		364 365
6 8/	98 8/				
6 8/	4 8/				366
7 8/	3 49 8/	3 8/	8/		367
5 8/	9 /				368
5 8/	7 8/	— 3 8/	1 8/	4 8/	69 37
8 8/	7 /	3 9 8/	46 /		
5 8/	/				37
7 8/	47 /	8 8/	4 36 /		37
8/	8/	6 8/	5 8/		373
4 8/	4 8/	6 8/			374 375
5 8/	(76 8/	3 8/			
3 /	7 8/	93 /	3 43 8/		376
45 8/	7 /	— 3 59 8/			3 7
8/	8/	69 8/	63 8/		3 8
8 8/	3 8/				379 38
8 8/	5 8/				
8/	97 /	7 8/	3 97 8/		38
8/	3 8/	6 58 8/	5 8/		38
8/					383
73 8/	4 94 8/				384 385
— 6 8/	3 8/	— 5 8/	8 7 8/		
5 8/	7 8/	— 6 67 8/			386
9 8/	/	61 8/	43 8/		387
3 /	97 8/				3 8
66 8/	9 8/	68 8/	8/		389 39
8 8/	7 /	7 8/			
— 3 8/	8/	9 8/			39
3 8/	5 /	4 8/			39
— 8/	8 /	— 3 39 8/			393
4 8/	8 8/	— 4 3 /			394 395
49 /	5 4 8/				
5 8/	99 /				396
6 8/	98 8/				397
7 /	3 8/				398
5 /	58 /				399 4

NUMERICAL LUNAR THEORY

SECTION X

SOLUTION OF THE EQUATIONS OF SECTION IX

Part 1.—General Remarks on the Steps of Solution.

Part 2.—Solution of the Equations (to No. 100 inclusive) which admit of Large Divisors.

Part 3.—Solution of the Equations (to No. 100 inclusive) with Small Divisors requiring different treatment.

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NUMERICAL LUNAR THEORY

SECTION X — SOLUTION OF THE EQUATIONS OF SECTION IX

Part 1 — General Remarks on the Steps of Solution

Each line of the equations of Section IX is an equation independent of every other equation, and (as the number of equations, after Nos 1 and 301, is the same as the number of unknown quantities) is competent to furnish a definite value for each unknown quantity But the great complexity of the equations makes it impracticable to solve them all by one general process, or even to restrict the mode of solution to one class of operations But some suggestions may be made, tending to limit the modes of operation and to assure the accuracy of every result

It seems proper first to call attention to the necessity of solving simultaneously the two subordinate equations in each line of the δ Equations (10) and (11) of Section IX The arguments of the great periodical terms in corresponding lines of the two equations are the same, the larger attached terms are similar, and the smaller attached terms are similar The two equations of each No must in fact be incorporated, in the ways which are most proper for obtaining the desired results, and the solutions must go on *pari passu* The form and nature of the solutions will depend in a most important degree on the adhesion to this principle

Next, it will be remarked that the equations, about 200 in number, are so connected successively, that, logically considered, all ought to be solved simultaneously As this is impracticable, we are compelled to treat each couplet (including the equations with the same No in δ Equation (10) and δ Equation δ (11)) separately from those in other couplets We cannot thus avoid the introduction of small multiples of the less important terms in each couplet, and we must carefully retain them, because it is by them that the successive connexion (to which we have just alluded) is really maintained We may rely on the results of other equations favourable for the very approximate numerical determination of those which are here the less important, and the result of the solution of the couple, thus supplemented, may be certainly accurate

It will, however, appear that the method, which is perfectly successful in the majority of cases, does (for reasons which will be assigned) fail in others But we are enabled in those cases to substitute a different method adapted to their peculiarities, which appears quite satisfactory

And, finally, it will be remarked that the mean excentricity of the Moon's orbit, and the mean inclination of its plane to the ecliptic, are elements in its original constitution which cannot really be inferred or corrected from theory And, therefore, adopting h_2 and k_{301} as the algebraical terms which best represent these elements, $\delta h_2 = 0$, and $\delta k_{301} = 0$ Apparently, an insignificant value is attributed to δh_2, which is to be rejected

The treatment of the terms of δ Equation (12) scarcely requires notice In each line, an approximate value of δk will be formed, by neglecting all terms after the first, and then this value, as well as δg and δh, is to be used for improving the numerical term in those lines in which it enters with a small co efficient

Part 2 — Solution of the Equations to No 100 inclusive, which admit of large divisors

The greater part of these equations may be solved by a simple and uniform process, in the accuracy of which much confidence may be placed Corresponding equations are to be used, one

derived from δ Equation (10), and the other from δ Equation (11) Thus (taking a couplet at hazard) we have for No 18,

From δ Equation (10)

$$0 = -924 + 5\ 97\ \delta g_{18} - 3\ 42\ \delta h_{18} - 0\ 34\ \delta g_{14} - 0\ 15\ \delta g_{06} + 0\ 14\ \delta k_{889} - 0\ 13\ \delta k_{355}$$

From δ Equation (11)

$$0 = -551 + 3\ 42\ \delta g_{18} - 2\ 95\ \delta h_{18} - 0\ 02\ \delta g_{3} - 0\ 10\ \delta g_{14} - 0\ 09\ \delta g_{06} + 0\ 02\ \delta h_{3}$$
$$+ 0\ 25\ \delta h_{14} + 0\ 07\ \delta h_{06} + 0\ 15\ \delta h_{38} - 0\ 15\ \delta k_{3.5}$$

The co-efficients of the terms following the three first in each equation are small Neglecting them in the first instance, we have two equations remaining, suitable for the determination of approximate values of δg_{18} and δh_{18} The form of solution is this—

Reference No 18	*Argument* $\overline{4D - 2l}$		
δ EQUATION (10)	[a] Excess -924	[b] Factor of δg $+5\ 97$	[c] Factor of δh $-3\ 42$
δ EQUATION (11)	[d] Excess -551	[e] Factor of δg $+3\ 42$	[f] Factor of δh $-2\ 95$
	$ce = -11\ 69$ $bf = -17\ 61$ $-ce+bf = -5\ 92$ = divisor	$dc = +1884$ $af = +2725$ $dc-af = -841$ = dividend for δg	$ae = -3160$ $db = -3289$ $ae-db = +129$ = dividend for δh
		$\delta g = +142$	$\delta h = -22$

The same process is followed for every No in the entire series, giving in each case values of its δg and δh, in most instances fairly approximate The approximate solution of the equations (to be mentioned shortly), giving the numerical values of δk, is very simple On examining all those values, it will be found that there are terms (δg_3, δg_{14}, &c, δh_3, &c, δk_{889}, &c) corresponding to those which were neglected in the treatment of No 18 (above) Substituting their numerical values, multiplying them by the small co efficients included in the equations No 18 of δ Equation (10) and δ Equation (11), and adding the products to the numerical terms (-924 and -551 respectively) of these two equations, we have two equations of the simplest character for two unknown quantities Re-solving these equations, we obtain very approximate values for δg_{18} and δh_{18}

The following Table contains the results of application of this process to the greater portion of the equations of Section IX The reasons for abandoning this process in reference to the remaining equations, and the results of applying a different process, will be given in the next sub-section

Solution of the Equations of Section IX

Part 2.—Investigation of the Numerical Values of δg and δh, for Terms admitting large Divisors

No	Argument	[a] Col 74 corrected	[b] Factor of δg	[c] Factor of δh	[d] Col 75, corrected	[e] Factor of δg	[f] Factor of δh	−ce + bf, divisor	dc − af, dividend for δg	ae − db, dividend for δh	δg	δh
1	o or nt											
4	2D	− 1005	+ 6 44	− 3 68	− 564	+ 3 68	− 3 43	− 8 547	− 1371	− 66	+ 160	+ 8
5	2l	+ 625	+ 6 95	− 3 94	+ 347	+ 3 95	− 3 93	− 11 75	+ 1089	+ 57	− 93	− 5
6	2D + l	− 782	+ 11 12	− 5 65	− 387	+ 5 66	− 8 08	− 57 871	− 4132	− 123	+ 71	+ 2
7	2D − S	− 7[illegible]4	+ 6 17	− 3 53	− 443	+ 3 54	− 3 16	− 7 001	− 882	− 7	+ 126	+ 1
13	3l	+ 340	+ 11 89	− 5 92	+ 166	+ 5 93	− 8 86	− 70 22	+ 2029	+ 42	− 29	− 1
14	4D − l	− 3024	+ 10 38	− 5 39	− 1551	+ 5 40	− 7 35	− 47 19	− 13566	− 231	+ 294	+ 5
17	2D + S	− 480	+ 6 72	− 3 83	− 267	+ 3 84	− 3 71	− 10 22	− 758	− 44	+ 74	+ 1
18	4D − 2l	− 1023	+ 5 97	− 3 42	− 587	+ 3 42	− 2 95	− 5 92	− 1010	+ 5	+ 170	− 1
20	2D + 2l	− 644	+ 17 76	− 7 62	− 283	+ 7 64	− 14 70	− 202 86	− 7311	+ 106	+ 36	− 1
21	2D + l − S	− 1050	+ 10 69	− 5 50	− 538	+ 5 51	− 7 66	− 51 55	− 5064	− 35	+ 99	+ 1
22	4D	− 3222	+ 16 76	− 7 36	− 1405	+ 7 37	− 13 71	− 175 54	− 33833	− 199	+ 193	+ 1
25	2l − S	− 424	+ 6 66	− 3 80	− 226	+ 3 80	− 3 65	− 9 87	− 689	− 106	+ 70	+ 11
27	D + l	+ 281	+ 6 69	− 3 81	+ 22	+ 3 82	− 3 68	− 10 06	+ 950	+ 926	− 94	− 92
28	2l + S	+ 231	+ 7 26	− 4 09	+ 128	+ 4 10	− 4 24	− 14 01	+ 455	+ 18	− 32	− 1
30	2D − 2S	− 255	+ 5 91	− 3 38	− 140	+ 3 39	− 2 90	− 5 68	− 267	− 37	+ 47	+ 7
33	4l	+ 57	+ 18 81	− 7 89	+ 11	+ 7 90	− 15 75	− 233 93	+ 811	+ [illegible]43	− 3	− 1
34	2D + l + S	+ 294	+ 11 55	− 5 80	+ 161	+ 5 81	− 8 51	− 64 59	+ 1568	− 152	− 24	+ 2
35	4D − l − S	− 1080	+ 9 97	− 5 24	− 517	+ 5 25	− 6 94	− 41 68	− 4786	− 516	+ 113	+ 12
36	3D − l	+ 301	+ 6 20	− 3 55	+ 157	+ 3 55	− 3 19	− 7 175	+ 403	+ 96	− 56	− 13
37	4D + l	− 1784	+ 25 12	− 9 33	− 553	+ 9 35	− 22 04	− 466 41	− 34160	− 2789	+ 73	+ 6
38	2D + 3l	− 482	+ 26 40	− 9 60	− 164	+ 9 61	− 23 30	− 522 86	− 9657	− 302	+ 18	+ 1
39	4D − 2l − S	− 279	+ 5 71	− 3 27	− 155	+ 3 27	− 2 70	− 4 72	− 246	+ 27	+ 52	− 6
40	3D	+ 398	+ 10 74	− 5 52	+ 152	+ 5 53	− 7 71	− 52 28	+ 2230	+ 568	− 43	− 11
41	2D + 2f − 2l	0	+ 6 53	− 3 73	− 2	+ 3 74	− 3 52	− 9 04	+ 7	+ 13	− 1	− 1
42	D + l + S	− 67	+ 6 99	− 3 96	− 35	+ 3 97	− 3 97	− 12 03	− 127	− 21	+ 11	+ 2
47	2D + 2l − S	− 526	+ 17 20	− 7 48	− 194	+ 7 49	− 14 14	− 187 18	− 5987	− 603	+ 32	+ 3
48	2D + 2f − l	− 2	+ 11 26	− 5 70	− 2	+ 5 71	− 8 23	− 60 12	− 5	+ 12	0	0
49	4D − S	− 1035	+ 16 21	− 7 21	− 363	+ 7 22	− 13 16	− 161 27	− 11367	− 1589	+ 70	+ 10
51	2f	+ 60	+ 7 05	− 3 99	+ 36	+ 4 01	− 4 04	− 12 48	+ 98	− 13	− 8	+ 1
52	2D − 2f − 2l	+ 24	+ 7 61	+ 4 26	− 18	− 4 26	− 4 59	− 16 78	+ 33	+ 35	− 2	− 2
53	2D − 4l	+ 86	+ 7 50	+ 4 21	− 46	− 4 22	− 4 48	− 15 83	+ 191	− 18	− 12	+ 1
54	4D − l + S	+ 156	+ 10 79	− 5 54	+ 130	+ 5 54	− 7 76	− 53 04	+ 491	− 539	− 9	+ 10
58	4D − 2l + S	+ 19	+ 6 23	− 3 56	+ 26	+ 3 57	− 3 21	− 7 29	− 32	− 94	+ 4	+ 13
60	3l − S	− 172	+ 11 45	− 5 77	− 74	+ 5 78	− 8 42	− 63 06	− 1021	− 147	+ 16	+ 2
61	3l + S	+ 126	+ 12 34	− 6 06	+ 12	+ 6 08	− 9 31	− 78 04	+ 1100	+ 618	− 11	− [illegible]
62	2D − 2f + 2l	− 15	+ 6 35	− 3 63	− 8	+ 3 64	− 3 33	− 7 93	− 21	− 4	+ 3	+ 1
63	D + 2l	+ 142	+ 11 50	− 5 78	+ 30	+ 5 79	− 8 46	− 63 82	+ 1026	+ 477	− 16	− 7
64	2D + l − 2S	− 175	+ 10 28	− 5 35	− 76	+ 5 36	− 7 25	− 45 85	− 562	− 157	+ 19	+ [illegible]
67	4D − 2f	+ 10	+ 5 88	− 3 37	+ 9	+ 3 37	− 2 87	− 5 52	− 1	− 19	0	+ [illegible]
68	5l	− 74	+ 27 70	− 9 86	− 40	+ 9 88	− 24 61	− 584 28	− 1427	+ 377	+ 2	− 1
72	3D + l	+ 110	+ 17 26	− 7 49	+ 14	+ 7 50	− 14 21	− 169 09	+ 1458	+ 583	− 5	− 3
73	3D − l + S	− 12	+ 6 47	− 3 70	− 9	+ 3 70	− 3 46	− 8 70	− 9	+ 14	+ 1	− 2
76	2f − S	+ 17	+ 6 76	− 3 84	+ 13	+ 3 85	− 3 74	− 10 50	+ 14	− 23	− 1	+ 2
77	2f + S	+ 43	+ 7 36	− 4 14	+ 18	+ 4 15	− 4 34	− 14 76	+ 112	+ 46	− 5	− 3
80	D + 2l + S	+ 7	+ 11 94	− 5 93	+ 7	+ 5 94	− 8 91	− 71 16	+ 20	− 4	0	+ 1
81	3D + S	− 18	+ 11 16	− 5 67	− 11	+ 5 68	− 8 13	− 58 53	− 84	+ 21	+ 1	0
83	2D + 2l + S	+ 189	+ 18 35	− 7 77	+ 64	+ 7 78	− 15 29	− 220 12	+ 2397	+ 296	− 11	− 1
85	4D + S	+ 198	+ 17 33	− 7 51	+ 88	+ 7 52	− 14 27	− 190 82	+ 2165	− 36	− 11	0
87	3D − S	0	+ 10 33	− 5 37	− 9	+ 5 38	− 7 30	− 46 52	+ 48	+ 93	− 1	− 2
90	2l − 2S	− 27	+ 6 38	− 3 05	− 15	+ 3 65	− 3 36	− 8 11	− 36	− 3	+ 4	0
91	2l + 2S	− 1	+ 7 57	− 4 24	0	+ 4 25	− 4 56	− 16 50	− 5	− 4	0	0
92	2D + 2S	+ 31	+ 7 02	− 3 98	+ 19	+ 3 98	− 4 00	− 12 24	+ 48	− 10	− 4	+ 1
93	2f + l	+ 5	+ 12 04	− 5 97	+ 1	+ 5 98	− 9 01	− 72 78	+ 39	+ 18	− 1	0
96	D + l − S	+ 39	+ 6 41	− 3 66	+ 16	+ 3 67	− 3 40	− 8 36	+ 74	+ 40	− 9	− 5

Part 3.—New Investigation of the Numerical Values of δg and δh, for Terms not admitting Large Divisors.

It will be convenient now to examine the constitution of the numbers employed in the last solutions. We will write the equations thus,

$$\begin{aligned} o &= a + b \cdot \delta g + c \cdot \delta h, \\ o &= d + e \cdot \delta g + f \cdot \delta h. \end{aligned}$$

It will be remembered that a and d are the Numerical Terms in the lines of Section IX. corresponding to the No. and Argument of the line in question; and that $(b, e,)$ (c, f) are co-efficients of (δg), (δh), in the same line; and that all these have been taken from the Product-Sheets, which have been formed by numerical development of the Modified Factorial Table in Section VIII.

From the equations written above, we obtain—

$$\begin{array}{llll}
o = ae & + be \cdot \delta g & + ce \cdot \delta h & \\
o = bd & + be \cdot \delta g & + bf \cdot \delta h & \\
\hline
o = (ae - bd) & + o & + (ce - bf) \cdot \delta h & \\
\hline\hline
o = af & + bf \cdot \delta g & + cf \cdot \delta h & \\
o = cd & + ce \cdot \delta g & + cf \cdot \delta h & \\
\hline
o = (af - cd) & + (bf - ce)\, \delta g + o & & \\
\hline\hline
\end{array}$$

Therefore—

$$\delta g = \frac{cd - af}{bf - ce}, \text{ or } = \frac{\frac{cd}{ef} - \frac{a}{e}}{\frac{b}{e} - \frac{c}{f}}, \text{ or } = \frac{\frac{d}{f} - \frac{a}{c}}{\frac{b}{c} - \frac{e}{f}},$$

$$\delta h = \frac{ae - bd}{bf - ce}, \text{ or } = \frac{\frac{a}{f} - \frac{bd}{ef}}{\frac{b}{e} - \frac{c}{f}}, \text{ or } = \frac{\frac{ae}{cf} - \frac{bd}{cf}}{\frac{b}{c} - \frac{e}{f}}.$$

The first of the forms for each element is the most convenient for use, but the second or third gives a clearer idea of the effect of special relations between the numbers employed.

If $b : c :: e : f$, or if $b : e :: c : f$, (which are equivalent comparisons of proportions), then the denominator $= o$, and the equations are useless.

If the actual proportions, in any case before us, are not exactly equal, but differ little from equality, then the results for δg and δh are large, and a trifling error in b, c, e, or f, will produce a very large error in the results for δg and δh.

This leads to the necessity of adopting, for these cases, a different process of solution, described in the following paragraphs:—

On examination it appears that forty-three of the equations following No. 7 (or more than two-fifths of the whole) are in the state described at the end of the last paragraph. And the

reason is this In developing the operations of Section VIII, terms of the various series taken from Sections II, III, &c, are multiplied by sin l or cos l, and produce terms whose arguments differ from the originals only by l, but which have the same proportion of co efficients for $+ l$ and $- l$ And when these are multiplied back again in the operations just described for finding δg and δh, so as to exhibit equations corresponding to the original argument, those which have been derived through $+ l$ and $- l$ will have the same proportion in their places in the two terms bf and ce Examination of the numerical operation will make this more clear

To illustrate the course which will now be pursued, we will take one pair of these terms, No 15, Argument S. They are the following,

$$\text{From } \delta' \text{ Equation (10)}, \; - 236 + 3\ 002\ \delta g_1 - 0\ 149\ \delta h_{15} = 0,$$
$$\text{From } \delta \text{ Equation (11)} \; - 9 + 0\ 149\ \delta g_1 - 0\ 006\ \delta h_1 = 0$$

On account of the similarity of proportions which I have mentioned (namely, the approximate equality of $\frac{149}{3\ 00}$ and $\frac{006}{149}$ I cannot treat them as separate equations And as I have no reason for presuming on the superior accuracy of either, I shall simply add them, and thus form the single equation,

$$- 245 + 3\ 151\ \delta g_{15} - 0\ 155\ \delta h_1 = 0$$

To separate the two inequalities δg_{15} and δh_1, I remark that summation of all the values of δg and δh (without regard of sign), derived in each case from the solution of each of the pairs of two equations connecting unknown quantities (exhibited in preceding pages, Part 2) shows that the errors of the equations are almost entirely derived from δg, and that we may take with sufficient exactness $\delta g = 25\ \delta h$ Assuming this proportion to apply generally to the equations now before us

$$- 245 + 3\ 151 \times 25 \times \delta h_1 - 0\ 155\ \delta h_{15} = 0,$$
$$\text{or } - 245 + 78\ 62\ \delta h_{15} = 0,$$
$$\delta h_1 = + 3\ 1, \; \delta g_1 = + 77$$

The treatment of the equations is not strictly accurate, but, in its practical result, I believe that it is scarcely inferior to the more complete investigation

Thus the following Table has been formed

Part 3—Investigation of the Numerical Values of δg and δh, for Terms not admitting large Divisors

No	Argument	δ Equation (10) = 0	δ Equation (11) = 0	Results δg	Results δh
2	l	$+ 366 + 3\ 99 \times \delta g - 1\ 97 \times \delta h$	$+ 198 + 1\ 98 \times \delta g - 0\ 99 \times \delta h$	$- 96\ 1$	$3\ 9$
3	$2D - l$	$- 1578 + 3\ 74 \times \delta g_3 - 1\ 71 \times \delta h_3$	$- 698 + 1\ 71 \times \delta g_3 - 0\ 71 \times \delta h_3$	$+ 425\ 3$	$+ 17\ 0$
8	$2D - l - S$	$+ 58 + 3\ 62 \times \delta g_8 - 1\ 56 \times \delta h_8$	$+ 35 + 1\ 56 \times \delta g_8 - 0\ 62 \times \delta h_8$	$- 18\ 3$	$- 0\ 7$
9	$l - S$	$- 1093 + 3\ 85 \times \delta g_9 - 1\ 83 \times \delta h_9$	$- 514 + 1\ 83 \times \delta g_9 - 0\ 84 \times \delta h_9$	$+ 288\ 3$	$+ 11\ 5$
10	D	$+ 639 + 3\ 86 \times \delta g_{10} - 1\ 83 \times \delta h_{10}$	$+ 292 + 1\ 83 \times \delta g_{10} - 0\ 86 \times \delta h_{10}$	$- 166\ 8$	$- 6\ 7$
11	$l + S$	$+ 45[illegible] + 4\ 14 \times \delta g_{11} - 2\ 12 \times \delta h_{11}$	$+ [illegible]32 + 2\ 1[illegible] \times \delta g_{11} - 1\ 14 \times \delta h_{11}$	$- 111\ 6$	$- 4\ 5$
12	$2f - l$	$- 15 + 4\ 04 \times \delta g_1 - 2\ 02 \times \delta h_1$	$+ 2 + 2\ 02 \times \delta g_1 - 1\ 03 \times \delta h_1$	$+ 2\ 2$	$+ 0\ 1$
15	S	$- 239 + 3\ 002 \times \delta g_1 - 0\ 149 \times \delta h_{15}$	$- 9 + 0\ 149 \times \delta g_1 - 0\ 006 \times \delta h_1$	$+ 75\ 8$	$+ 3\ 2$
16	$2D - l + S$	$- 1643 + 3\ 88 \times \delta g_{16} - 1\ 86 \times \delta h_{16}$	$- 786 + 1\ 86 \times \delta g_{16} - 0\ 87 \times \delta h_{16}$	$+ 431\ 4$	$+ 17\ 3$
19	$2D - 2l$	$+ 241 + 3\ 02 \times \delta g_{19} + 0\ 26 \times \delta h_{19}$	$- 26 - 0\ 26 \times \delta g_{19} - 0\ 02 \times \delta h_{19}$	$- 77\ 7$	$- 3\ 1$

Part 3.—Investigation of the Numerical Values of δg and δh, for Terms not admitting large Divisors— completed

No	Argument	δ Equation (10) = 0	δ Equation (11) = 0	Results δg	Results δh
23	$D \quad +S$	− 19 + 4 00 × δg − 1 99 × δh	− 9 + 1 99 × δg − 1 00 × δh	+ 4 8	+ 0 2
24	$2D-2f$	− 33 + 3 02 × δg + 0 31 × δh	+ 3 − 0 32 × δg − 0 02 × δh	+ 11 1	+ 0 4
26	$2D \quad -3l$	+ 20[illegible] + 4 27 × δg + 2 24 × δh	− 103 − 2 24 × δg − 1 [illegible]6 × δh	− 47 8	− 1 9
29	$2D-2f-l$	+ 15 + 4 33 × δg + 2 29 × δh	− 7 − [illegible]9 × δg − 1 32 × δh	− 3 8	− 0 [illegible]
31	$2D-2f+l$	− 24 + 3 70 × δg_{31} − 1 66 × δh_{31}	− 10 + 1 66 × δg_{31} − 0 70 × δh_{31}	+ 6 5	+ 0 3
32	$2D \quad -l-2S$	− 35 + 3 51 × δg − 1 41 × δh	− 15 + 1 41 × δg − 0 50 × δh	+ 9 5	+ 0 4
43	$l-2S$	− 54 + 3 71 × δg_{43} − 1 68 × δh_{43}	− 25 + 1 68 × δg_{43} − 0 71 × δh_{43}	+ 14 9	+ 0 6
44	$2D \quad -l+2S$	+ 67 + 4 0[illegible] × δg_{44} − 2 00 × δh_{44}	+ 35 + 2 01 × δg_{44} − 1 02 × δh_{44}	− 16 9	− 0 7
45	$2D \quad -2l-S$	+ 61 + 3 04 × δg + 0 41 × δh	− 8 − 0 41 × δg + 0 04 × δh	− 20 0	− 0 8
46	$2S$	− 36 + 3 02 × δg_{46} − 0 30 × δh_{46}	− 3 + 0 30 × δg_{46} − 0 02 × δh_{46}	+ 11 8	+ 0 5
50	$l+2S$	+ [illegible] + 4 31 × δg_{50} − 2 27 × δh_{50}	+ 2 + 2 27 × δg_{50} − 1 30 × δh_{50}	− 0 9	0 0
55	$D \quad -l$	− 39 + 3 00 × δg + 0 13 × δh	+ 3 − 0 13 × δg — × δh	+ 1[illegible] 5	+ 0 5
56	$D-2f$	+ 2 + 4 18 × δg_{56} + [illegible] 15 × δh_{56}	+ 1 − 2 16 × δg_{56} − 1 17 × δh_{56}	− 1 4	− 0 1
57	$2D-2f \quad -S$	+ 9 + 3 05 × δg_{57} + 0 46 × δh_{57}	+ 1 − 0 46 × δg_{57} − 0 05 × δh_{57}	− 3 8	− 0 2
59	$2D \quad -2l$	− 75 + 4 1[illegible] × δg + 2 10 × δh_{59}	+ 39 − 2 11 × δg − 1 12 × δh	+ 17 6	+ 0 7
65	$3D \quad -2l$	+ 32 + 3 65 × δg − 1 58 × δh	+ 14 + 1 58 × δg − 0 63 × δh	− 9 0	− 0 4
66	$4D \quad -3l$	− 15 + 3 53 × δg_{66} − 1 41 × δh_{66}	− 8 + 1 45 × δg_{66} − 0 53 × δh_{66}	+ 4 7	+ 0 2
69	$2D-2f-l-S$	+ [illegible] + 4 50 × δg_{69} + [illegible] 44 × δh_{69}	− 1 − 2 44 × δg_{69} − 1 50 × δh_{69}	− 0 5	0 0
70	$4D-2f-l$	+ 7 + 3 49 × δg_{70} − 1 39 × δh_{70}	+ 4 + 1 40 × δg_{70} − 0 49 × δh_{70}	− [illegible] 3	− 0 1
71	$2D \quad -3l-S$	+ 24 + 4 44 × δg_{71} + 2 35 × δh_{71}	− 13 − 2 39 × δg_{71} − 1 14 × δh_{71}	− 5 3	− 0 [illegible]
74	$2D-2f+l-S$	+ 2 + 3 58 × δg_{74} − 1 51 × δh_{74}	+ 1 + 1 51 × δg_{74} − 0 58 × δh_{74}	− 0 6	0 0
75	$D \quad -S$	+ 17 + 3 73 × δg_{75} − 1 69 × δh_{75}	− 2 + 1 69 × δg_{75} − 0 72 × δh_{75}	− 2 8	− 0 1
78	$2f-l-S$	− 6 + 3 89 × δg_{78} − 1 87 × δh_{78}	− 3 + 1 88 × δg_{78} − 0 89 × δh_{78}	+ 1 6	+ 0 1
79	$2f-l+S$	+ 12 + 4 20 × δg_{79} − 2 17 × δh_{79}	+ 8 + 2 17 × δg_{79} − 1 19 × δh_{79}	− 3 2	− 0 1
82	$3D-2f$	− 5 + 3 59 × δg_{82} − 1 53 × δh_{82}	− 2 + 1 53 × δg_{82} − 0 59 × δh_{82}	+ 1 4	+ 0 1
84	$2D \quad -2l+S$	+ 23 + 3 00 × δg_{84} + 0 12 × δh_{84}	− 2 − 0 12 × δg_{84} — × δh_{84}	− 7 3	− 0 3
86	$D \quad -2l+S$	+ 10 + 3 97 × δg_{86} + 1 96 × δh_{86}	− 5 − 1 96 × δg_{86} − 0 97 × δh_{86}	− 2 4	− 0 1
88	$2D \quad -3l+S$	+ 2[illegible] + 4 11 × δg_{88} + 2 09 × δh_{88}	− 1[illegible] − 2 09 × δg_{88} − 1 10 × δh_{88}	− 4 9	− 0 2
89	$2D-2f-l+S$	+ 5 + 4 16 × δg_{89} + 2 14 × δh_{89}	− 4 − 2 14 × δg_{89} − 1 15 × δh_{89}	− 0 5	0 0
94	$2D-2f \quad +S$	− 2 + 3 00 × δg_{94} + 0 17 × δh_{94}	0 − 0 16 × δg_{94} — × δh_{94}	+ 0 7	0 0
95	$2D-2f+l+S$	0 + 3 83 × δg_{95} − 1 81 × δh_{95}	+ 1 + 1 81 × δg_{95} − 0 83 × δh_{95}	0 0	0 0
97	$2f-3l$	− 5 + 3 94 × δg_{97} + 1 92 × δh_{97}	+ 7 − 1 93 × δg_{97} − 0 94 × δh_{97}	− 1 0	0 0
98	$D \quad -2S$	+ 3 + 3 60 × δg_{98} − 1 54 × δh_{98}	+ 2 + 1 54 × δg_{98} − 0 60 × δh_{98}	− 1 0	0 0
99	$D-2f \quad +S$	− 2 + 4 02 × δg_{99} + 2 00 × δh_{99}	+ 1 − 2 01 × δg_{99} − 1 0[illegible] × δh_{99}	+ 0 5	0 0
100	$D \quad +2S$	− 1 + 4 16 × δg_{100} − [illegible] 14 × δh_{100}	0 + 2 14 × δg_{100} − 1 16 × δh_{100}	+ 0 2	0 0

Part 4.—Examination of the Magnitude of Terms of Long Period

It appears desirable to show, at least in one instance, the effect that we may expect to find generally in the co-efficients of terms of long period, produced by the length of their periods. There are several instances in the series of terms following No. 100 (at which our complete investigations have stopped), which merit attention, especially Nos. 101, 102, 108, 147, 202. Of these, I select No. 102, argument $|D-l+S|$, as the term of longest period; its "movement" in the table of Section II, Part 2, being + 0 0084513, and its period in orbital revolutions $\frac{1\ 0000000}{0\ 0084513}$ or 118 325, or 8 76 years nearly.

The Modified Factorial Table of Section VIII is adapted to this inquiry. We are to search out, as substitutes for H in that table, every argument (including No. 102) which, com-

bined with the terms on the left side of the table, will produce No 102, the multiplications there indicated are to be performed, every term of the form cos $|\overline{D - l + S}|$ or sin $|\overline{D - l + S}|$ is to be retained, and the sum of all is to equal 0

With one exception (namely, the co-efficient of $\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}|\overline{3D - 2l + S}|$, an exceedingly small term, not sensible in parallax or longitude, required here for combination with $|\overline{2D - l}|$) every term required in the investigation is to be found in Section II or Section IX

We proceed now with the calculations, arranged in tabular form—

Examination of the Magnitude of a Term of Long Period

Argument of Term under consideration = $D - l + S$ No 102, Equation (10)

No of Line of Factorial Table	2 Left side of Modified Factorial Table	3 Right side of Modified Factorial Table (omitting m and m)	4 Product of Columns 2 and 3 (omitting ineffective terms)	5 1, m or m corresponding to Column 3	Product of Columns 4 and 5
1		$\delta q \times \cos H$		+1	
	+2 996	$\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$	+2 996 $\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$	+1	+2 996 × $\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$
	−0 054 cos l	$\{+5 \times \cos \lvert\overline{D+S}\rvert$	−0 135 × cos $\lvert\overline{D-l+S}\rvert$	+1	−0 135 × cos $\lvert\overline{D-l+S}\rvert$
		$-2 \times \cos \lvert\overline{D-2l+S}\rvert$	+0 054 × cos $\lvert\overline{D-l+S}\rvert$	+1	+0 054 × cos $\lvert\overline{D-l+S}\rvert$
	−0 007 cos $\lvert\overline{2D-l}\rvert$	$-3 \times \cos \lvert\overline{D-S}\rvert$	+0 011 × cos $\lvert\overline{D-l+S}\rvert$	+1	+0 011 × cos $\lvert\overline{D-l+S}\rvert$
	−0 024 cos $\lvert\overline{-D}\rvert$	$\{+1 \times \cos \lvert\overline{3D-l+S}\rvert$	−0 012 × cos $\lvert\overline{D-l+S}\rvert$	+1	−0 012 × cos $\lvert\overline{D-l+S}\rvert$
		$-9 \times \cos \lvert\overline{D+l-S}\rvert$	+0 108 × cos $\lvert\overline{D-l+S}\rvert$	+1	+0 108 × cos $\lvert\overline{D-l+S}\rvert$
	+0 005 cos $\lvert\overline{-l}\rvert$	$+16 \times \cos \lvert\overline{D+l+S}\rvert$	+0 040 × cos $\lvert\overline{D-l+S}\rvert$	+1	+0 004 × cos $\lvert\overline{D-l+S}\rvert$
2		$\delta g \times \sin H$		$-m$	
	−0 214 sin l	$\{+5 \times \sin \lvert\overline{D+S}\rvert$	−0 535 × cos $\lvert\overline{D-l+S}\rvert$	−1 0000	+0 535 × cos $\lvert\overline{D-l+S}\rvert$
		$-2 \times \sin \lvert\overline{D-2l+S}\rvert$	−0 214 × cos $\lvert\overline{D-l+S}\rvert$	+0 9831	−0 211 × cos $\lvert\overline{D-l+S}\rvert$
	−0 032 sin $\lvert\overline{2D-l}\rvert$	$-3 \times \sin \lvert\overline{D-S}\rvert$	+0 048 × cos $\lvert\overline{D-l+S}\rvert$	−0 8504	−0 041 × cos $\lvert\overline{D-l+S}\rvert$
	−0 053 sin $\lvert\overline{2D}\rvert$	$\{+1 \times \sin \lvert\overline{3D-l+S}\rvert$	−0 026 × cos $\lvert\overline{D-l+S}\rvert$	−1 8588	+0 048 × cos $\lvert\overline{D-l+S}\rvert$
		$-9 \times \sin \lvert\overline{D+l-S}\rvert$	+0 238 × cos $\lvert\overline{D-l+S}\rvert$	−1 8419	−0 439 × cos $\lvert\overline{D-l+S}\rvert$
3		$\delta q \times \cos H$		$-m$	
	−1 005	$\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$	−1 005 $\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$	−0 00007	+0 000 × $\delta q_{10} \times \cos \lvert\overline{D-l+S}\rvert$
	+0 163 cos l	$\{+5 \times \cos \lvert\overline{D+S}\rvert$	+0 408 × cos $\lvert\overline{D-l+S}\rvert$	−1 0000	−0 408 × cos $\lvert\overline{D-l+S}\rvert$
		$-2 \times \cos \lvert\overline{D-2l+S}\rvert$	−0 163 × cos $\lvert\overline{D-l+S}\rvert$	−0 966	+0 158 × cos $\lvert\overline{D-l+S}\rvert$
	+0 028 cos $\lvert\overline{2D-l}\rvert$	$-3 \times \cos \lvert\overline{D-S}\rvert$	−0 042 × cos $\lvert\overline{D-l+S}\rvert$	−0 723	+0 031 × cos $\lvert\overline{D-l+S}\rvert$
	+0 021 cos $\lvert\overline{-D}\rvert$	$\{+1 \times \cos \lvert\overline{3D-l+S}\rvert$	+0 010 × cos $\lvert\overline{D-l+S}\rvert$	−3 457	−0 038 × cos $\lvert\overline{D-l+S}\rvert$
		$-9 \times \cos \lvert\overline{D+l-S}\rvert$	−0 094 + cos $\lvert\overline{D-l+S}\rvert$	−3 394	+0 372 × cos $\lvert\overline{D-l+S}\rvert$

Part 4—Examination of the Magnitude of a Term of Long Period—continued

Argument of Term under consideration $= D - l + S$, No 102, Equation (10)

No of Line of Factorial Table	2 Left side of Modified Factorial Table	3 Right side of Modified Factorial Table (omitting m and m)	4 Product of Columns 2 and 3 (omitting ineffective terms)	5 1, m or m corresponding to Column 3	Product of Columns 4 and 5
4	$(-0{\cdot}0005 \sin l)$ $-0{\cdot}003 \sin \overline{2D-l}$ $+0{\cdot}017 \sin \overline{2D}$	$\delta h \times \sin H$ $0 \times \sin \overline{D-S}$ $\{-2 \times \sin \overline{3D-l+S}$ $\{-5 \times \sin \overline{D+l-S}$	$0{\cdot}000$ $0{\cdot}000$ $-0{\cdot}017 \times \cos \overline{D-l+S}$ $-0{\cdot}042 \times \cos \overline{D-l+S}$	$+1$ $+1$ $+1$	$-0{\cdot}017 \times \cos \overline{D-l+S}$ $-0{\cdot}042 \times \cos \overline{D-l+S}$
5	$-1{\cdot}989$ $(+0{\cdot}0006 \cos l)$ $+0{\cdot}003 \cos \overline{2D-l}$ $-0{\cdot}009 \cos \overline{2D}$	$\delta h \times \cos H$ $\delta h_{10} \times \cos \overline{D-l+S}$ $0 \times \cos \overline{D-S}$ $\{-2 \times \cos \overline{3D-l+S}$ $\{-5 \times \cos \overline{D+l-S}$	$-1{\cdot}989\, \delta h_{10} \times \cos \overline{D-l+S}$ $0{\cdot}000$ $0{\cdot}000$ $+0{\cdot}009 \times \cos \overline{D-l+S}$ $+0{\cdot}023 \times \cos \overline{D-l+S}$	$+m$ $+0{\cdot}00845$ $+1{\cdot}8588$ $+1{\cdot}8419$	$-0{\cdot}017 \times \delta h_{10} \times \cos \overline{D-l+S}$ $+0{\cdot}017 \times \cos \overline{D-l+S}$ $+0{\cdot}042 \times \cos \overline{D-l+S}$

Argument of Term under consideration $= D - l + S$, No 102 Equation (11)

No of Line of Factorial Table	2 Left side of Modified Factorial Table	3 Right side of Modified Factorial Table (omitting m and m)	4 Product of Columns 2 and 3 (omitting ineffective terms)	5 1, m or m corresponding to Column 3	Product of Columns 4 and 5
11	$-{\cdot}107 \sin l$ $-{\cdot}016 \sin \overline{2D-l}$ $-{\cdot}028 \sin \overline{2D}$	$\delta q \times \cos H$ $\{+5 \times \cos \overline{D+S}$ $\{-2 \times \cos \overline{D-2l+S}$ $-3 \times \cos \overline{D-S}$ $\{+1 \times \cos \overline{3D-l+S}$ $\{+9 \times \cos \overline{D+l-S}$	$+0{\cdot}368 \times \sin \overline{D-l+S}$ $+0{\cdot}107 \times \sin \overline{D-l+S}$ $+0{\cdot}024 \times \sin \overline{D-l+S}$ $+0{\cdot}014 \times \sin \overline{D-l+S}$ $+0{\cdot}126 \times \sin \overline{D-l+S}$	1 1 1 1 1 1	$+0{\cdot}268 \times \sin \overline{D-l+S}$ $+0{\cdot}107 \times \sin \overline{D-l+S}$ $+0{\cdot}024 \times \sin \overline{D-l+S}$ $+0{\cdot}014 \times \sin \overline{D-l+S}$ $+0{\cdot}126 \times \sin \overline{D-l+S}$
12	$-1{\cdot}992$ $+{\cdot}109 \cos l$ $+{\cdot}022 \cos \overline{2D-l}$ $+{\cdot}006 \cos \overline{2D}$	$\delta q \times \sin H$ $\delta q_{10} \times \sin \overline{D-l+S}$ $\{0 \times \sin \overline{D+S}$ $\{0 \times \sin \overline{D-2l+S}$ $0 \times \sin \overline{D-S}$ $\{-2 \times \sin \overline{3D-l+S}$ $\{-5 \times \sin \overline{D+l-S}$	$-1{\cdot}992\, \delta q_{10} \times \sin \overline{D-l+S}$ $0{\cdot}000$ $0{\cdot}000$ $0{\cdot}000$ $-0{\cdot}006 \times \sin \overline{D-l+S}$ $+0{\cdot}015 \times \sin \overline{D-l+S}$	$-m$ $-0{\cdot}00845$ $-1{\cdot}0000$ $+0{\cdot}9831$ $-0{\cdot}8504$ $-1{\cdot}8588$ $-1{\cdot}8419$	$+0{\cdot}0168\, \delta q_{10} \times \sin \overline{D-l+S}$ $+0{\cdot}011 \times \sin \overline{D-l+S}$ $-0{\cdot}0$[illegible]$7 \times \sin \overline{D-l+S}$
13	$-0{\cdot}002$ $-{\cdot}005 \cos l$ $+{\cdot}003 \cos \overline{2D-l}$ $+{\cdot}017 \cos \overline{2D}$	$\delta h \times \sin H$ $\delta h_{10} \times \sin \overline{D-l+S}$ $\{0 \times \sin \overline{D+S}$ $\{0 \times \sin \overline{D-2l+S}$ $0 \times \sin \overline{D-S}$ $\{-2 \times \sin \overline{3D-l+S}$ $\{-5 \times \sin \overline{D+l-S}$	$-0{\cdot}002\, \delta h_{10} \times \sin \overline{D-l+S}$ $0{\cdot}000$ $0{\cdot}000$ $0{\cdot}000$ $-0{\cdot}017 \times \sin \overline{D-l+S}$ $+0{\cdot}042 \times \sin \overline{D-l+S}$	1 1 1 1 1 1 1	$-0{\cdot}002\, \delta h_{10} \times \sin \overline{D-l+S}$ $-0{\cdot}017 \times \sin \overline{D-l+S}$ $+0{\cdot}042 \times \sin \overline{D-l+S}$

Part 4—Examination of the Magnitude of a Term of Long Period—completed

Argument of Term under consideration $= D - l + S$ No 102 Equation (11)

No of Line of Factorial Table	2 Left side of Modified Factorial Table	3 Right side of Modified Factorial Table (omitting m and m)	4 Product of Columns 2 and 3 (omitting ineffective terms)	5 1 m or m corresponding to Column 3	Product of Columns 4 and 5
14		$\delta h \times \cos H$		m	
	$+ \cdot 108 \sin l$	$+5 \times \cos \lvert D+S \rvert$	$-0\cdot270 \times \sin \lvert D-l+S \rvert$	$+1\cdot0000$	$-0\cdot270 \sin \lvert D-l+S \rvert$
		$-2 \times \cos \lvert \overline{D-2l+S} \rvert$	$-0\cdot108 \times \sin \lvert D-l+S \rvert$	$-0\cdot9831$	$+0\cdot106 \sin \lvert D-l+S \rvert$
	$+ \cdot 016 \sin \lvert \overline{2D-l} \rvert$	$-3 \times \cos \lvert D-S \rvert$	$-0\cdot024 \times \sin \lvert D-l+S \rvert$	$+0\cdot8504$	$0\cdot000 \sin \lvert D-l+S \rvert$
	$+ \cdot 027 \sin \lvert 2D \rvert$	$+1 \times \cos \lvert \overline{3D-l+S} \rvert$	$-0\cdot014 \times \sin \lvert D-l+S \rvert$	$+1\cdot8588$	$0\cdot026 \sin \lvert D-l+S \rvert$
		$-9 \times \cos \lvert \overline{D+l-S} \rvert$	$-0\cdot122 \times \sin \lvert D-l+S \rvert$	$+1\cdot8419$	$-0\cdot225 \sin \lvert D-l+S \rvert$
15		$\delta h \times \sin H$		m	
	$+1\cdot006$	$\delta h_{10} \times \sin \lvert \overline{D-l+S} \rvert$	$+1\cdot006\, \delta h_{10} \times \sin \lvert D-l+S \rvert$	$-0\cdot0003$	$0\cdot000$
	$- \cdot 108 \cos l$	$0 \times \sin \lvert D+S \rvert$	$0\cdot000$	$-1\cdot000$	
		$0 \times \sin \lvert \overline{D-2l+S} \rvert$	$0\cdot000$	$-0\cdot966$	
	$- \cdot 019 \cos \lvert \overline{2D-l} \rvert$	$0 \times \sin \lvert D-S \rvert$	$0\cdot000$	$-0\cdot723$	
	$- \cdot 015 \cos \lvert 2D \rvert$	$-2 \times \sin \lvert \overline{3D-l+S} \rvert$	$+0\cdot015 \times \sin \lvert D-l+S \rvert$	$-3\cdot457$	$-0\cdot052 \sin \lvert D-l+S \rvert$
		$-5 \times \sin \lvert \overline{D+l-S} \rvert$	$-0\cdot038 \times \sin \lvert D-l+S \rvert$	$-3\cdot394$	$+0\cdot126 \times \sin \lvert D-l+S \rvert$

Collecting and summing the terms from these two tables, we find the following equations —

$$\text{From Equation (10)}, + 2\cdot996 \times \delta g_{102} - 0\cdot017 \times \delta h_{102} - \cdot37 = 0,$$
$$\text{From Equation (11)}, + 0\cdot017 \times \delta g_{102} - 0\cdot002 \times \delta h_{102} - \cdot187 = 0$$

And we have now to decide on the process to be adopted for solution of these equations

We are here in a difficulty precisely similar to that in Part 3 of this Section We have before us two equations which, physically, are exceedingly unequal (the physical units being the same) If we assume both to be accurate, we obtain, for solutions, numbers extravagantly large If we make, in the smaller equation, petty numerical changes, such as could well be adopted as consistent with the possibility of small errors, we produce enormous changes in magnitude of the results, or even change of sign The best combination of the equations which it is possible to make appears to be their simple sum, or

$$+ 3\cdot013 \times \delta g_{102} - 0\cdot019\, \delta h_{2} - \cdot224 = 0$$

But we must have another equation or another condition, and here I propose, as in Part 3 of this Section, to assume that $\delta g = 25 \times \delta h$ This changes the equation into $3\cdot012\, \delta g_{102} - \cdot224 = 0$, $\delta g_{102} = + \cdot 074$, $\delta h_{102} = + \cdot 003$

The value of δg_{102} represents a term in the direction of radius vector, whose measure is the length of nearly $1''\cdot5$ on the Moon's orbit, it is totally insensible to observation The value of δh_{102} represents a term in longitude $= 0''\cdot06$ nearly, it is too small to be seen

Part 5.—On the possibility of introducing Secular Terms.

I have not discovered any form of Secular Terms which can satisfy the equations applying to the Moon's co-ordinates, unless the motions of the Sun or the Planets are affected by some secular cause unrecognized in the preceding investigations.

To mathematicians who desire to examine this question, I would suggest that great simplicity is introduced by omitting all terms depending in any way on f, l, or S, but that the retention of the simple arguments D, $|\overline{2D}|$, $|\overline{3D}|$, &c. is indispensable, as they are necessary elements in the expression of movement in a system in which the Moon's motions are disturbed by the attraction of the Sun.

Part 6.—Final expressions for the Moon's Horizontal Equatoreal Parallax and Longitude, on assumption of Spherical Earth and Invariable Solar Orbit.

The co-efficients under the heading "value of $\frac{a}{r}$ further corrected," which correspond to radius 10000000, are converted into co-efficients corresponding to the Sexagesimal Equivalent of Parallax, by the multiplier $\frac{3442''.33}{10000000}$.

RESULTS OF THE ENTIRE INVESTIGATION OF LUNAR ECLIPTIC INEQUALITIES.

No.	Term: Argument	Coefficient of Cosine of Argument: Assumed Value of $\frac{a}{r}$ (Column 1), corrected for the numbers in Section V.	Value of $\frac{a}{r}$ further corrected for δg.	Corresponding Equatoreal Horizontal Parallax, Sexagesimal.	Coefficient of Sine of Argument: Assumed Value of v corrected for δh.	Converted Value of v, Sexagesimal.
				′ ″		° ″
1	0 or nt	+ 1.0000004	+ 1.0000000	+ 57 2.33		
2	l	+ 545077	+ 544981	+ 3 6.51		
3	$2D - l$	+ 99813	+ 100238	+ 34.31	+ 1097565	+ 6 17 19.0
4	$2D$	+ 52320	+ 82480	+ 28.23	+ 222353	+ 1 16 26.4
5	$2l$	+ 29794	+ 29701	+ 10.17	+ 114895	+ 39 29.9
6	$2D + l$	+ 8950	+ 9021	+ 3.10	+ 9311	+ 3 12.1
7	$2D - S$	+ 5462	+ 5608	+ 1.92	+ 8017	+ 2 45.4
8	$2D - l - S$	+ 4226	+ 4206	+ 1.44	+ 10001	+ 3 26.3
9	$l - S$	+ 3075	+ 3363	+ 1.15	+ 7153	+ 2 27.5
10	D	− 2740	− 2907	− .99	− 6176	− 2 7.4
11	$l + S$	− 2663	− 2775	− .95	− 5322	− 1 49.8
12	$2f - l$	− 2084	− 2062	− .71	− 1917	− 39.5
13	$3l$	+ 1845	+ 1816	+ .63	+ 1752	+ 36.1
14	$4D - l$	+ 1447	+ 1741	+ .58	+ 1863	+ 38.4
15	S	− 1248	− 1169	− .40	− 32419	− 11 8.7
16	$2D - l + S$	− 1107	− 676	− .24	− 1412	− 29.1
17	$2D + S$	− 954	− 880	− .30	− 1188	− 24.5
18	$4D - 2l$	+ 907	+ 1077	+ .37	+ 1478	+ 30.5
19	$2D - 2l$	− 809	− 887	− .31	+ 10248	+ 3 31.4
20	$2D + 2l$	+ 791	+ 827	+ .29	+ 697	+ 14.4

Part 6 —Results of the entire Investigation of Lunar Ecliptic Inequalities continued

TERM		COEFFICIENT OF COSINE OF ARGUMENT			COEFFICIENT OF SINE OF ARGUMENT	
No	Argument	Assumed Value of $\frac{a}{r}$, Column 1, corrected for the numbers in Section V	Value of $\frac{a}{r}$ further corrected for δy	Corresponding Equatorial Horizontal Parallax, Sexagesimal	Assumed Value of v corrected for δh	Converted Value of v, Sexagesimal
				′ ″		′ ″
21	$2D + l - S$	+ 575	+ 674	+ 23	+ 708	+ 1[illegible] 6
22	$4D$	+ 57[illegible]	+ 765	+ [illegible]7	+ 675	+ 13 9
23	$D + S$	+ 439	+ 444	+ 15	+ [illegible]59	+ 18 3
24	$2D - 2f$	− 3[illegible]0	− 309	− 10	+ 2[illegible]6	+ 55 [illegible]
25	$2l - S$	+ 301	+ 371	+ 12	+ 477	+ 9 8
26	$2D - 3l$	− 296	− 314	− 11	+ 616	+ 1[illegible] 1
27	$D + l$	− 284	− 378	− 13	− 507	1[illegible] 5
28	$2l + S$	− 268	− 300	− 10	− 3[illegible]1	7 7
29	$2D - 2f - l$	− 239	− 243	− 05	+ 9	+ 2
30	$2D - 2S$	+ 2[illegible]	+ 269	+ 09	+ 397	+ 8 2
31	$2D - 2f + l$	− 146	− 139	− 05	− [illegible]09	6 4
32	$2D - l - 2S$	+ 13[illegible]	+ 14[illegible]	+ 05	+ 363	+ 7 5
33	$4l$	+ 1[illegible]1	+ 118	+ 04	+ 94	+ 1 9
34	$2D + l + S$	− 115	− 142	− 05	− 14	2 9
35	$4D - l - S$	+ 87	+ 20[illegible]	+ 0	+ 20	+ 4 5
36	$3D - l$	− 58	− 114	− 04	− 159	− 3 3
37	$4D + l$	+ 54	+ 127	+ 04	+ 96	+ [illegible] 0
38	$2D + 3l$	+ 51	+ 69	+ 02	+ 52	+ 1 1
39	$4D - 2l - S$	+ 46	+ 98	+ 03	+ 125	+ 2 6
40	$3D$	+ 46	+ 3		+ 1[illegible]	+ 4
41	$2D + 2f - 2l$	− 43	− 44	− 02	− 27	− 6
42	$D + l + S$	+ 39	+ 50	+ 02	+ 6	+ 1 3
43	$l - 2S$	+ 39	+ 54	+ 02	+ 121	+ 2 5
44	$2D - l + 2S$	− 38	− 55	− 0[illegible]	− 1[illegible]9	− 2
45	$2D - 2l - S$	− 37	− 57	− 0[illegible]	+ 419	+ 8 6
46	$2S$	− 36	− 24	− 01	− 362	− 7 5
47	$2D + 2l - S$	+ 35	+ 67	+ 02	+ 57	+ 1 2
48	$2D + 2f - l$	− 3	− 32	− 01	− [illegible]5[illegible]	− 9 4
49	$4D - S$	+ 33	+ 103	+ 04	+ 91	+ 1 9
50	$l + 2S$	− 30	− 31	− 01	− 56	− 1 2
51	$2f$	− 27	− 35	− 01	− 19955	− 6 51 6
52	$2D - 2f - 2l$	− 24	− 26	− 01	− [illegible]	− 6
53	$2D - 4l$	− [illegible]4	− 36	− 01	+ 16	+ 9
54	$4D - l + S$	− 22	− 31	− 01	− 31	− 6
55	$D - l$	+ 2[illegible]	+ 35	+ 01	− 90[illegible]	− 19 6
56	$D - 2f$	+ 22	+ [illegible]1	+ 01	− [illegible]	6
57	$2D - 2f - S$	− [illegible]0	− 24	− 01	+ 106	+ [illegible]
58	$4D - 2l + S$	− 20	− 16	− 01	− [illegible]0	− 4
59	$D - 2l$	+ 20	+ 38	+ 01	− 73	− 1 5
60	$3l - S$	+ 18	+ 34	+ 01	+ 33	+ 7
61	$3l + S$	− 18	[illegible]32	− 01	− 33	− 7
62	$2D - 2f + 2l$	− 19	− 16	− 01	− 21	− 4
63	$D + 2l$	− 19	− 35	− 01	− 35	− 7
64	$2D + l - 2S$	+ 15	+ 34	+ 01	+ 36	+ [illegible]
65	$3D - 2l$	− 16	− 25	− 01	− 58	− 1 2

Part 6 —Results of the entire Investigation of Lunar Ecliptic Inequalities—completed

TERM		COEFFICIENT OF COSINE OF ARGUMENT			COEFFICIENT OF SINE OF ARGUMENT	
No	Argument	Assumed Value of $\frac{a}{r}$, Column 1, corrected for the numbers in Section V	Value of $\frac{a}{r}$ further corrected for δg	Corresponding Equatorial Horizontal Parallax Sexagesimal	Assumed Value of v corrected for δh	Converted Value of v Sexagesimal
				′ ″	″	° ′ ″
66	$4D - 3l$	+ 12	+ 17	+ 01	+ 47	+ 1 0
67	$4D - 2f$	+ 12	+ 12		0	0
68	$5l$	+ 8	+ 10		+ 5	+ 1
69	$2D - 2f - l - S$	− 9	− 9		− 2	0
70	$4D - 2f - l$	+ 9	+ 7		+ 16	+ 3
71	$2D - 3l - S$	− 8	− 13		+ 24	+ 5
72	$3D + l$	+ 7	− 1		− 1	0
73	$3D - l + S$	+ 8	+ 9		+ 12	+ 2
74	$2D - 2f + l - S$	− 7	− 8		− 18	− 4
75	$D - S$	− 6	− 9		− 27	− 6
76	$2f - S$	+ 6	+ 5		− 2	0
77	$2f + S$	+ 6	− 2		+ 17	+ 4
78	$2f - l - S$	+ 6	+ 8		+ 6	+ 1
79	$2f - l + S$	− 6	− 9		− 4	− 1
80	$D + 2l + S$	+ 6	+ 6		+ 5	+ 1
81	$3D + S$	+ 5	+ 6		+ 7	+ 1
82	$3D - 2f$	− 6	− 5		− 13	− 3
83	$2D + 2l + S$	− 5	− 16	− 01	− 14	− 3
84	$2D - 2l + S$	+ 5	− 2		+ 91	+ 1 9
85	$4D + S$	− 5	− 16	− 01	− 14	− 3
86	$D - 2l + S$	− 4	− 6		+ 13	+ 3
87	$3D - S$	+ 5	+ 4		+ 3	+ 1
88	$2D - 3l + S$	+ 4	− 1		+ 3	+ 1
89	$2D - 2f - l + S$	+ 4	+ 4		0	0
90	$2l - 2S$	+ 3	+ 7		+ 8	+ 2
91	$2l + 2S$	− 3	− 3		− 3	− 1
92	$2D + 2S$	− 3	− 7		− 6	− 1
93	$2f + l$	− 3	− 4		− 2187	− 45 1
94	$2D - 2f + S$	+ 3	+ 4		− 70	− 1 4
95	$2D - 2f + l + S$	+ 3	+ 3		+ 5	+ 1
96	$D + l - S$	+ 3	− 6		− 9	− [illegible]
97	$2f - 3l$	− 2	− 3		+ 2	0
98	$D - 2S$	+ 2	+ 1		+ 2	0
99	$D - 2f + S$		+ 1		0	0
100	$D + 2S$	− 1	− 1		− 2	0

Part 7 —Final expressions for the Moon's latitude, on assumption of Spherical Earth and Invariable Solar Orbit

The equations relating to the correction of the Moon's vertical distance from the plane of the ecliptic (which is sensibly the same as the correction of the Moon's latitude) are so simple, that the whole of the operations for the solution of the equations in the third division of Section IX by ascertaining and applying the values of δh, can be included in the two small tables following

The primary co efficient of latitude is adopted from Delaunay

Part 7.—Results of the entire Investigation of Inequalities of Lunar Latitude

No	Argument	—	Factor of δl	δl
301	f			0
302	$f + l$	+ 13	− 2 97	− 4
303	$f - l$	+ 37	+ 1 01	+ 37
304	$2D - f$	+ 1	+ 0 [illegible]92	+ 3
305	$2D + f - l$	− 40	− 2 46	+ 16
306	$2D - f - l$	− 11	+ 0 987	− 11
307	$2D + f$	− 12	− 7 14	+ 2
308	$f + 2l$	+ 3	− 7 91	0
309	$D - f + l$	0	− [illegible] 37	0
310	$f - 2l$	− 3	+ 0 049	− 61
311	$2D - f - S$	+ 1	+ 0 413	+ [illegible]
312	$2D - f - 2l$	− 1	− 0 [illegible]85	+ 4
313	$2D + f + l$	+ 12	− 13 78	− 1
314	$2D - f - S$	+ 3	+ 0 159	+ 19
315	$2D + f - l - S$	+ 11	− [illegible] 19	− 5
316	$2D + f - S$	− 1	− 6 72	0
317	$2D - f - l - S$	− 1	+ 0 960	− 1
318	$4D - f - l$	− 8	− 1 90	+ 4
319	$f + l - S$	− 31	− [illegible] 65	− 12
320	$f + S$	+ 6	− 0 156	− 38
321	$3f$	+ 1	− [illegible] 06	0
322	$f - l + S$	− 34	+ 1 00	− [illegible]
323	$D + f$	+ 22	− [illegible] 71	− 8
324	$f + l + S$	+ 15	− 3 27	− 5
325	$f - l - S$	+ 2	+ 1 01	+ 2
326	$f - S$	− 3	+ 0 145	− 21
327	$D - f$	− 5	+ 1 002	− 5
328	$f + 3l$	+ 3	− 14 8[illegible]	0
329	$4D - f$	− 2[illegible]	− 6 26	+ 4
330	$4D + f - l$	− 91	− 1[illegible] 76	+ 7
331	$3f - l$	+ 14	− 3 07	− 5
332	$4D + f - 2l$	− 37	− 6 10	+ 6
333	$[illegible]D - 3f$	+ 1	− 0 342	− 3
334	$2D - f + 2l$	+ 11	− 6 99	− 2
335	$[illegible]D + f - l + S$	− 52	− 2 75	+ 19
336	$2D - f + l - S$	+ 1	− [illegible] 10	0
337	$2D + f - 2l$	+ 7	+ 0 249	+ 25
338	$f - 3l$	− 158	− [illegible] 87	+ 55
339	$2D + f + 2l$	− 12	− 22 40	+ 1
340	$2D - f - 3l$	− 3	− 3 52	+ 1
341	$2D - f + S$	− 50	− 7 57	+ 7
342	$2D - f - l + S$	+ 18	+ 1 003	+ 18
343	$2D - f - 2S$	0	+ 0 522	0
344	$2D + f + l - S$	− 47	− 13 21	+ 4
345	$4D + f$	− 152	− 21 13	+ 7
346	$3f + l$	+ 1	− 15 02	0
347	$2D - f + l + S$	− 27	− 2 65	+ 10
348	$D - f + S$	0	+ 1 01	0
349	$D + f + S$	− 3	− 3 00	+ 1
350	$f + 2l - S$	− 28	− 7 47	+ 4

No	Argument	—	Factor of δl	δl
351	$2D - f - l - S$	+ [illegible]	− 0 [illegible]	[illegible]
352	$D + f + l$	+ 31	[illegible] 5	
353	$f + 2l + S$	+ 6	[illegible] 3	[illegible]
354	$D - f + l$	− 8	+ [illegible]	[illegible]
355	$4D - f - 2l$	− 1	+ 0 [illegible]	
356	$4D - f - l - S$	[illegible]	1 65	
357	$4D - f + l$	6[illegible]	1 [illegible]	[illegible]
358	$4D - f - S$	0	5 [illegible]	
359	$D - f - l$	+ 1	0 [illegible]	
360	$2D + f - S$	− 5	6 31	1
361	$3D - f$	− 1	− [illegible]	
362	$[illegible]D - 3f - l$	− 5	3 [illegible]	[illegible]
363	$f + 2l - S$	+ 1	0 [illegible]	[illegible]
364	$2D + f + l - S$	+ 3	1 [illegible]	
365	$f - l + S$	0	+ 0 [illegible]	
366	$2D - 3f + l$	[illegible]	+ [illegible]	[illegible]
367	$3D - f - l$	[illegible]	+ [illegible]	[illegible]
368	$f + 4l$	+ [illegible]	− [illegible]	[illegible]
369	$2D - f - l - 2S$	[illegible]	+ [illegible]	[illegible]
370	$4D + f - l - S$	15	1 [illegible]	[illegible]
371	$2D + 3f - l$	+ 9	[illegible]	[illegible]
372	$D + f - 3l$	− [illegible]	+ [illegible]	[illegible]
373	$D + f + l + S$	3	[illegible]	
374	$4D - f - l + S$	1	[illegible]	
375	$[illegible]D + f - [illegible]l - S$	1		
376	$3D + f - l$	+ 15	6 [illegible]	
377	$[illegible]D + f + l$	− 1	31 [illegible]	[illegible]
378	$2D - f + 3l$	+ 3	[illegible]	
379	$[illegible]D + 3f$	[illegible]	[illegible]	
380	$D + f - l$	+ [illegible]	+ 0 [illegible]	[illegible]
381	$3f - 2l$	0	[illegible]	
382	$2D + f + 3l$	65	32 [illegible]	
383	$2D - f + l - S$	+ 9	− [illegible]	[illegible]
384	$[illegible]D - f + 2S$	0 52	+ [illegible]	[illegible]
385	$3f + [illegible]l$	[illegible]	[illegible]	
386	$2D - f - 4l$	− 5	[illegible]	[illegible]
387	$[illegible]D - f + S$	8	[illegible]	[illegible]
388	$[illegible]D + f - S$	− 9	− [illegible]	
389	$D + f + l + S$	− 5	[illegible]	[illegible]
390	$2D + f + 2l - S$	− 13	− 21 6[illegible]	[illegible]
391	$f - [illegible]S$	+ 1	[illegible]	[illegible]
392	$f + 2S$	0	− [illegible]	
393	$f + l - 2S$	− 2	− 2 40	[illegible]
394	$f + l + 2S$	− 3	[illegible]	[illegible]
395	$f + 3l - S$	+ 6	− 14 [illegible]	
396	$f + 3l + S$	− 15	− 15 4	[illegible]
397	$f - l - 2S$	0	+ [illegible]	
398	$f - l + 2S$	0	+ 0 98	[illegible]
399	$f - 3l - S$	+ 1	− 3 1[illegible]	
400	$f - 3l + S$	− 3	− 2 [illegible]	[illegible]

Part 7 —Final expressions for the Moon's Latitude

No	Argument	Co efficient of Sine of Argument, Corrected for δl	Sexagesimal Equivalent ° ″	No	Argument	Co efficient of Sine of Argument Corrected for δk	Sexagesimal Equivalent ° ′ ″
301	f	+ 895027	+ 5 7 41 3	351	$2D - f - 2l - S$	+ 31	+ 6
302	$f + l$	+ 48974	+ 16 50 2	352	$D + f + l$	− 36	− 7
303	$f - l$	+ 48506	+ 16 40 5	353	$f + 2l + S$	− 31	− 6
304	$2D - f$	+ 30237	+ 10 23 7	354	$D - f + l$	− 75	− 1 5
305	$2D + f - l$	+ 9678	+ 3 19 6	355	$4D - f - 2l$	+ 26	+ 5
306	$2D - f - l$	+ 8056	+ 2 46 2	356	$4D - f - l - S$	+ 28	+ 6
307	$2D + f$	+ 5684	+ 1 57 2	357	$4D - f + l$	+ 27	+ 6
308	$f + 2l$	+ 3006	+ 1 2 0	358	$4D - f - S$	+ 19	+ 4
309	$2D - f + l$	+ 1618	+ 33 4	359	$D - f - l$	− 25	− 5
310	$f - 2l$	− 1602	+ 33 0	360	$2D + f - 2S$	+ 18	+ 4
311	$2D - f - S$	+ 1440	+ 29 7	361	$3D - f$	− 16	− 3
312	$2D - f - 2l$	+ 748	+ 15 4	362	$2D - 3f - l$	+ 16	+ 3
313	$2D + f + l$	+ 732	+ 15 1	363	$f - 2l - S$	+ 6	+ 1
314	$2D - f - S$	− 572	− 11 8	364	$2D + f - l - 2S$	+ 13	+ 3
315	$2D + f - l - S$	+ 431	+ 8 9	365	$f - 2l + S$	− 15	− 3
316	$2D + f - S$	+ 387	+ 8 0	366	$2D - 3f + l$	− 14	− 3
317	$2D - f - l - S$	+ 361	+ 7 4	367	$3D - f - l$	− 19	− 4
318	$4D - f - l$	+ 321	+ 6 6	368	$f + 4l$	+ 14	+ 3
319	$f + l - S$	+ 328	+ 6 8	369	$2D - f - l - 2S$	+ 13	+ 3
320	$f + S$	− 352	− 7 3	370	$4D + f - l - S$	+ 17	+ 4
321	$3f$	− 306	− 6 3	371	$2D + 3f - l$	− 13	− 3
322	$f - l + S$	− 295	− 6 1	372	$2D + f - 3l$	− 20	− 4
323	$D + f$	− 267	− 5 5	373	$2D + f + l + S$	− 12	− 2
324	$f + l + S$	− 260	− 5 4	374	$4D - f - l + S$	− 10	− 2
325	$f - l - S$	+ 245	+ 5 1	375	$4D + f - 2l - S$		
326	$f - S$	+ 220	+ 4 5	376	$3D + f - l$	− 10	− 2
327	$D - f$	− 236	− 4 9	377	$4D + f + l$	+ 9	+ 2
328	$f + 3l$	+ 195	+ 4 0	378	$2D - f + 3l$	+ 7	+ 1
329	$4D - f$	+ 182	+ 3 8	379	$2D + 3f$	− 7	− 1
330	$4D + f - l$	+ 144	+ 3 0	380	$D + f - l$	+ 23	+ 5
331	$3f - l$	− 136	− 2 8	381	$3f - 2l$	+ 6	+ 1
332	$4D + f - 2l$	+ 116	+ 2 4	382	$2D + f + 3l$	+ 8	+ 2
333	$2D - 3f$	+ 104	+ 2 1	383	$2D - f + 2l - S$	+ 5	+ 1
334	$2D - f + 2l$	+ 103	+ 2 1	384	$2D - f + 2S$	− 39	− 8
335	$2D + f - l + S$	− 68	− 1 4	385	$3f + 2l$	− 6	− 1
336	$2D - f + l - S$	+ 85	+ 1 8	386	$2D - f - 4l$	+ 7	+ 1
337	$2D + f - 2l$	− 56	− 1 2	387	$4D - f + S$	− 5	− 1
338	$f - 3l$	− 23	− 0 5	388	$4D + f - S$	+ 5	+ 1
339	$2D + f + 2l$	+ 75	+ 1 5	389	$D + f + l + S$	+ 6	+ 1
340	$2D - f - 3l$	+ 72	+ 1 5	390	$2D + f + 2l - S$	+ 6	+ 1
341	$2D - f + S$	− 60	− 1 2	391	$f - 2S$	+ 6	+ 1
342	$2D - f - l + S$	− 40	− 8	392	$f + 2S$	− 3	− 1
343	$2D - f - 2S$	+ 52	+ 1 1	393	$f + l - 2S$	+ 6	+ 1
344	$2D + f + l - S$	+ 56	+ 1 2	394	$f + l + 2S$	− 2	0
345	$4D + f$	+ 79	+ 1 6	395	$f + 3l - S$	+ 3	+ 1
346	$3f + l$	− 49	− 1 0	396	$f + 3l + S$	− 2	0
347	$2D - f + l + S$	− 30	− 6	397	$f - l - 2S$	+ 3	+ 1
348	$D - f + S$	+ 39	+ 8	398	$f - l + 2S$	− 3	− 1
349	$D + f + S$	+ 40	+ 8	399	$f - 3l - S$	+ 1	0
350	$f + 2l - S$	+ 40	+ 8	400	$f - 3l + S$	0	0

With these Tables terminates the discussion of the magnitude of co-efficients of separate terms as depending on the assumptions of Spherical Earth and Undisturbed Position of the Solar Orbit

SECTION X — SOLUTION OF THE EQUATIONS OF SECTION IX

Part 8 — Remarks on the Correction of the Orbital Elements

In the preceding steps of the Solution of the Individual Equations derived from the mass of equations which are virtually collected in Equation (10), Equation (11), Equation (12), I have, in order to diminish the great complexity of the case treated the equations by supposing that they might be separated into two classes (namely, those which apply to Individual Coefficients of Inequalities and those which show the effects of errors of General Orbital Elements applying to all), and I have tacitly assumed that these two classes might be treated separately without material error

The class of Individual Coefficients has been discussed at great length

There remains now, to be examined the class of Orbital Elements Of these, as applying to the Plane of the Ecliptic (the movement parallel to that plane being not sensibly affected by small errors in the terms of latitude), there is, in perfect accuracy of language, only one error, namely, that of the movement of argument of elliptic inequality, although it will be convenient to use, for its investigation, the supposition of two inequalities, with arguments of the same period, one applying to radius vector, the other applying to longitude

For this purpose I have taken account of all the terms as far as No 25 omitting all in which the argument is merely a multiple of l, and also omitting all in which l does not appear And I have divided these adopted terms into two classes, distinguished by the sign of l in the argument For the numerical terms uncorrected I have adopted the leading numbers in Section IX, Parts 1 and 2, changing the signs of all, to show the correction required For the corrections to g and h (δg and δh), I have referred to Section X, Parts 2 and 3

Examination of the Discordance of Results, as connected with the sign of l in the Arguments

(1) When the sign of l in the Argument is positive

No	Sign and Multiple of l in the Argument	Correction required by Numerical Term of Equation (10)		Correction required by Numerical Term of Equation (11)		Correction found for g for Equation (10)		Correction found for h for Equation (11)	
6	+ 1	+ 647		+ 385		+ 71		+ 2	
9	+ 1	+ 1080		+ 506		+ 288		+ 12	
11	+ 1		− 451		− 235		− 112		− 5
20	+ 2	+ 550		+ 269		+ 36			− 1
21	+ 1	+ 940		+ 514		+ 99		+ 1	
25	+ 2	+ 352		+ 196		+ 70		+ 11	
Sum	+ 8	+ 3118		+ 1635		+ 453		+ 20	
Mean	1	+ 390		+ 204		+ 57		+ 2	

(2.) When the sign of l in the Argument is negative.

No.	Sign and Multiple of l in the Argument.	Correction required by Numerical Term of Equation (10).		Correction required by Numerical Term of Equation (11).		Correction found for q for Equation (10).		Correction found for h for Equation (11).	
3	− 1	+ 1563		+ 726		+ 425		+ 18	
8	− 1		− 71		− 39		− 18	0	− 1
12	− 1	+ 17		+ 0		+ 2	9	0	
14	− 1	+ 2774		+ 1496		+ 294		+ 5	
16	− 1	+ 1634		+ 784		+ 431		+ 17	
18	− 2	+ 924		+ 551		+ 170			− 1
19	− 2		− 238	+ 26			− 78		− 4
Sum	− 9	+ 6603		+ 3544		+ 1226		+ 34	
Mean	1	+ 734		+ 394		+ 136		+ 4	
General Mean		+ 562		+ 299		+ 96		+ 3	
General Mean reducing all to the negative sign of l		+ 172		+ 95		− 61		+ 1	

A similar course may be followed with regard to the inequalities of Lunar Latitude. The argument upon which all others are formed is f, and, as the primary value of f is arbitrary, the only way in which we can examine inaccuracies (on the broad scale) in the application of it, is, by comparing the mass of results in which the sign of f is + with the mass of results in which the sign of f is −. In the following table I omit terms depending simply on f and $3f$.

No.	Multiples of f +	Multiples of f −	Terms of Equation (12.) +	Terms of Equation (12.) −	Terms of δk +	Terms of δk −
302	1			11	3 7	
303	1			5		4 9
304		1		18		62 1
305	1		69			28 1
306		1		45		46 0
307	1		60			8 4
308	1			25	3 2	
309		1		12	5 1	
310	1			12		240
311		1		13		31 4
312		1	10			35 0
313	1		64			4 6
314		1		4		25 0
315	1		4			1 8

No.	Multiples of f +	Multiples of f −	Terms of Equation (12.) +	Terms of Equation (12.) −	Terms of δl +	Terms of δl −
316	1		50			7 4
317		1	1		1 0	
318		1		71	37 4	
319	1		38			14 2
320	1		15			98 0
322	1		14		14 0	
323	1			21	7 8	
324	1			17	5 2	
325	1			7	6 9	
326	1		7		51 2	
327		1	7		7 0	
328	1			21	1 4	
329		1		87	13 9	
330	1		104			8 1

It does not, however, appear possible, without extensive calculations, to offer any check of this class on the relation between the coefficients in the series for $\frac{a}{r}$ and those in the series for v, excepting that given by the general solution for δg and h at the beginning of Section X., and in this respect the present theory, at the point where it now stops, might seem defective. I believe, however, that on examination it will be found satisfactory.

Part 9.—Consideration of Theories.

I may now express my opinion on the two forms of treatment (Delaunay's and that of the present work) considered above, and on the course which, in my judgment, it would be best to adopt in future Lunar Theories.

I regard the contents of the present volume as a work, not so much of investigation as of criticism. And in this light I think that such a work, properly carried out, may be useful. But its results ought to be exhibited, not only in checking the coefficients of longitude and latitude (as above), but also so as to separate the effects of errors of the coefficients of current time in the arguments of the leading inequalities of excentricity and latitude. I cannot now hope personally, as I could have desired, to complete these steps.

I have been led to my undertaking by a belief that the approximation to the values of numerical terms of a secondary or tertiary place, by series of powers of the primary algebraical coefficient, is not perfectly trustworthy. The convergence of terms which M. Delaunay has exhibited is slow. It is easy to show in simple algebraical formulæ how this character may be given to results, which, if at once treated numerically, possess undoubted accuracy, but which by being involved in a process of symbols, may be rendered inaccurately divergent.

I am very sensible of the beauty of algebraical treatment in every step. Nevertheless, I consider that the best prospect for resultant accuracy is to be sought in algebraical treatment of numerical terms, step by step, always maintaining all the simple products, of each algebraic form derived from the last substitution, by the sum of all the numerical coefficients (without restriction of orders) collected from that last substitution. The treatment of the terms connected with excentricity and inclination (in the use of a' cos $n't$ + b' sin $n't$ for the excentric term in $\frac{a}{r}$, and a'' cos $n''t + b''$ sin $n''t$ for the excentric term in v) will require cautions which have not hitherto been necessary. Similar remarks apply to the latitude.

With this terminates my work on what is usually considered Lunar Theory.

I now proceed to consider the terms which depend on foreign elements:—the Figure of the Earth, and the Disturbances of the Solar Orbit.

NUMERICAL LUNAR THEORY.

SECTION XI.

TERMS PRODUCED BY OBLATENESS OF THE EARTH

NUMERICAL LUNAR THEORY

SECTION XI.—TERMS PRODUCED BY OBLATENESS OF THE EARTH

Let the axis of x be drawn from the center of the earth towards the first point of Cancer, the axis of y towards the first point of Libra, and the axis of z towards the North pole of the Ecliptic. The axis of the earth will be included in the plane which passes through the axes of x and z, being inclined from the positive part of the axis of z towards the positive part of the axis of x, and will make, with the axis of z, the angle ω the obliquity of the ecliptic nearly 23° 27′.

For expression of the Moon's co-ordinates and the forces which act on the Moon, we shall begin by referring all to a new system of co-ordinates, X being in the plane of the Earth's equator, where intersected by the planes xz, Y the same as y, and Z in the Earth's axis. Then for the Moon's co-ordinates,

$$\begin{aligned} X &= \cos\omega \cdot x - \sin\omega \cdot z \\ Y &= y \\ Z &= \sin\omega \cdot x + \cos\omega \cdot z \end{aligned}$$

Now, by the theory of the Heterogeneous Earth (see the author's Mathematical Tracts, 4th Edition, Cambridge, 1836, in which Maclaurin's Theory is strictly employed), putting E for the entire mass of the oblate earth, c the polar semi axis, $c(1+e)$ the equatorial semi axis, m the ratio of equatorial centrifugal force to gravity, also r for the Moon's distance from the Earth's center:—

$$\text{Force in X} = \mathrm{E}\left\{-\frac{X}{r^3} + \left(e - \frac{m}{2}\right) \times \frac{c^2}{r^7} \times \{-r^2 + 5Z^2\} \times X\right\}$$

$$\text{Force in Y} = \mathrm{E}\left\{-\frac{Y}{r^3} + \left(e - \frac{m}{2}\right) \times \frac{c^2}{r^7} \times \{-r^2 + 5Z^2\} \times Y\right\}$$

$$\text{Force in Z} = \mathrm{E}\left\{-\frac{Z}{r^3} + \left(e - \frac{m}{2}\right) \times \frac{c^2}{r^7} \times \{-3r^2 + 5Z^2\} \times Z\right\}$$

The numerical value of e is sensibly $\frac{1}{300} = 0.003333$, that of m is $\frac{1}{289} = 0.0034602$, therefore the value of $e - \frac{m}{2} = +0.001603$, or $= +\frac{1}{623.8}$.

The first terms of the expressions for the three forces represent, when taken together, the simple gravitational attraction of the mass E, as collected at the center of the earth, without any allusion to oblate form. It is intended (in this Section) to investigate the effect of oblateness by reference only to the forces which may be considered as added on to the ordinary forces of gravity, in the same manner as in other parts of the theory of disturbing forces. Therefore, we may now omit the first terms. And we may consider E = 1, and we may, for the present, omit

the general multiplier $+\frac{1}{623.8}\cdot\frac{c}{r^7}$. Then, substituting for X, Y, Z, their values given above, we obtain,

$$\text{Oblateness-force in X} = \left\{-r^2 + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\right\} \times (\cos\omega\,.\,x - \sin\omega\,.\,z)$$

$$\text{Oblateness-force in Y} = \left\{-r^2 + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\right\} \times y$$

$$\text{Oblateness-force in Z} = \left\{-3r^2 + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\right\} \times (\sin\omega\,.\,x + \cos\omega\,.\,z)$$

Now, by ordinary transfer of the direction of forces,

Oblateness-force in x = $+\cos\omega\times$ oblateness-force in X $+\sin\omega\times$ oblateness-force in Z
Oblateness-force in y = oblateness-force in Y
Oblateness-force in z = $-\sin\omega\times$ oblateness-force in X $+\cos\omega\times$ oblateness-force in Z

And then, by inserting in these the values given in the preceding lines,

Oblateness-force in $x = -2\sin\omega\,.\,r^2(\sin\omega\,.\,x + \cos\omega\,.\,z) - r^2x + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\,x$
Oblateness-force in $y = -r^2y + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\,y$
Oblateness-force in $z = -2\cos\omega\,.\,r^2(\sin\omega\,.\,x + \cos\omega\,.\,z) - r^2z + 5(\sin\omega\,.\,x + \cos\omega\,.\,z)^2\,z$

We now prepare for our proposed method of solution. Use ρ, for the length of the projection of r upon the plane of the ecliptic, or the hypotenuse of the triangle whose sides are x and y, and v, for the angle between x and ρ, or the 'Geocentric Longitude of the Moon $-\frac{\pi}{2}$.' Also, put λ for the Moon's Geocentric latitude. Then ρ will $= r\cos\lambda$. And—

Oblateness-force in ρ = $+\cos v\times$ oblateness-force in x $+\sin v\times$ oblateness-force in y

Oblateness-force transversal to ρ, in the direction of accelerating the orbital motion $= -\sin v\times$ oblateness-force in x $+\cos v\times$ oblateness-force in y

Or—

Oblateness-force Radial in Ecliptic =

$$-2\sin\omega\,.\,\frac{r^2x}{\rho}(\sin\omega\,.\,x + \cos\omega\,.\,z) - r^2\rho + 5\rho(\sin\omega\,.\,x + \cos\omega\,.\,z)^2$$

Oblateness-force Transversal in Ecliptic =

$$+2\sin\omega\,.\,\frac{r^2y}{\rho}(\sin\omega\,.\,x + \cos\omega\,.\,z)$$

Oblateness-force Normal to Ecliptic =

$$-2\cos\omega\,.\,r^2(\sin\omega\,.\,x + \cos\omega\,.\,z) - r^2z + 5z(\sin\omega\,.\,x + \cos\omega\,.\,z)^2$$

Re-introducing now the general multiplier $+0.001603\,\frac{c}{r^7}$, and remarking that $\omega = 23^\circ\ 27'$ nearly, sine $\omega = 0.39795$, cos $\omega = 0.91741$, also $\frac{c}{r^7} = \frac{c}{a}\cdot\frac{a^2}{r^7}$ = (sine of moon's mean horizontal polar parallax)$^2\ \frac{a}{r^7}$ = $(\sin 56'\ 51'')^2\ \frac{a}{r^7}$, and assuming $a = 1$, we obtain the following expressions with numerical coefficients, for the oblateness-forces just found. The first column of figures contains the logarithmic factors of the several terms produced by expansion of the

formulæ above, it will be remarked that in each of them the first numerical term is positive, but the large negative correction −10.0 is to be attached to it. As regards the second column the numbers are all to be multiplied by 10^{-10}. The arguments have been changed by application of the formulæ $x = \rho \cos v$, $y = \rho \sin v$.

(A.) General Multiplier (included in all the following expressions) } = + [3.6[illegible] − 10.0] × [illegible] + [illegible] [illegible]

Oblateness forces, Radial in Ecliptic							
(B)	=	− [3.14250 − 10.0]	a [illegible] ρ	-10^{-10}	1358.1	1 [illegible] ρ	cos [illegible]
(C)		− [3.50523 − 10.0]	a [illegible] ρ	-10^{-10} ×	3200.6	1 [illegible] ×	cos [illegible]
(D)	=	− [3.64181 − 10.0]	a [illegible] ρ	10^{-10}	4383.4	1 [illegible] ρ	
(E)	=	+ [3.54044 − 10.0]	a [illegible] ρ	+ 10^{-10}	3470.9	1 [illegible]	cos [illegible]
(F)		+ [4.20420 − 10.0]	a [illegible] ρ	+ 10^{-10}	16003.0	1 [illegible] ρ	cos [illegible]
(G)	=	+ [4.26590 − 10.0]	a [illegible] ρ	+ 10^{-10}	18446.0	1 [illegible] ρ	
Oblateness forces, Transversal in Ecliptic							
(H)	=	+ [3.14250 − 10.0]	a [illegible] ρ	+ 10^{-10} ×	1388.1	1 [illegible] ρ	cos [illegible] sin [illegible]
(I)	=	+ [3.505[illegible]3 − 10.0]	a [illegible] ρ	+ 10^{-10} ×	3[illegible]0.6	1 [illegible]	sin [illegible]
Oblateness forces, Normal to Ecliptic							
(J)	=	− [3.50523 − 10.0]	a [illegible]	-10^{-10}	3200.6	1 [illegible]	cos [illegible]
(K)	=	− [3.86796 − 10.0]	a [illegible]	-10^{-10} ×	7378.4 ×	1 [illegible]	
(L)	=	− [3.64181 − 10.0]	a [illegible]	-10^{-10} ×	4383.4	1 [illegible]	
(M)	=	+ [3.54044 − 10.0]	a [illegible]	+ 10^{-10} ×	3470.9 ×	1 [illegible] × ρ	cos [illegible]
(N)	=	+ [4.20420 − 10.0]	a [illegible]	+ 10^{-10} ×	16003.0	1 [illegible] × ρ ×	cos [illegible]
(P)	=	+ [4.26590 − 10.0]	a [illegible]	+ 10^{-10} ×	18446.0 ×	1 [illegible] ×	[illegible]

And these expressions are to be converted (by operations to be hereafter described) into formulæ depending on the general argument H, and on l, [2 D − l], &c., connected with H

We now proceed to treat these numbers by reference to the equations obtained in page 10.

In the equations (4), (5), (6), of page 10, put ρ for $r \cos l$ and P, T, Z, for the disturbing forces — parallel to ρ, ecliptic-transversal to ρ, and normal to the ecliptic. And consider ρ as represented by the sum of two terms, $R + \delta\rho$, v by $V + \delta v$, and l by $L + \delta l$, of which R, V, L, would satisfy the equations deprived of their perturbation terms, and $R + \delta\rho$, $V + \delta v$, $L + \delta l$, will satisfy the equations with the perturbation terms, $\delta\rho$, δv, δl, being extremely small. The factors of $\delta\rho$, δv, δl, in the functions of $R + \delta\rho$, &c., will be formed by the ordinary differential formulæ.

First, to produce equation (4), of page 10. In the last term, (ρf) represents the entire force in the direction of ρ, and therefore (see page 12) it is $= -\frac{E+\mu}{r^2}\cos l + P$, or $= -\frac{E+\mu}{\rho^2}\cos^3 l + P$, or, sensibly (page 11) and using 1 for a, $(\rho f) = -\frac{1}{\rho}\cos^3 l + P$. Thus the equation (4) becomes—

$$+\frac{1}{2}\frac{d^2(\rho^2)}{dt^2} - \left(\frac{d\rho}{dt}\right)^2 - \rho^2\left(\frac{dv}{dt}\right)^2 + \frac{1}{\rho}\cos^3 l - P\rho = 0$$

And, substituting $R + \delta\rho$, and $V + \delta v$, to the first power of $\delta\rho$ and δv,—

$$+\frac{1}{2}\frac{d^2(\rho^2)}{dt^2} = +\frac{1}{2}\frac{d^2(R^2 + 2R\,\delta\rho)}{dt^2} = \left\{\begin{array}{l} +\frac{1}{2}\frac{d^2(R^2)}{dt^2} + \frac{d^2R}{dt^2}\delta\rho \\ + 2\frac{dR}{dt}\frac{d\,\delta\rho}{dt} + R\frac{d^2\,\delta\rho}{dt^2} \end{array}\right\}$$

$$-\left(\frac{d\rho}{dt}\right)^2 = -\left(\frac{dR}{dt} + \frac{d\,\delta\rho}{dt}\right)^2 = -\left(\frac{dR}{dt}\right)^2 - 2\frac{dR}{dt}\frac{d\,\delta\rho}{dt}$$

$$-\rho^2\left(\frac{dv}{dt}\right)^2 = -(R+\delta\rho)^2\left(\frac{dV}{dt} + \frac{d\,\delta v}{dt}\right)^2 = \left\{\begin{array}{l} -R^2\left(\frac{dV}{dt}\right)^2 - 2R^2\left(\frac{dV}{dt}\right)\frac{d\,\delta v}{dt} \\ \qquad - 2R\left(\frac{dV}{dt}\right)^2\delta\rho \end{array}\right\}$$

$$+\frac{1}{\rho}\cos^3 l = +\left\{\begin{array}{l}\left(\frac{1}{R} - \frac{\delta\rho}{R^2}\right)\times \\ (\cos^3 L - 3\cos^2 L\sin L\,\delta l)\end{array}\right\} = \left\{\begin{array}{l} +\frac{1}{R}\cos^3 L - \frac{3}{R}\cos^2 L \\ \sin L\,\delta l - \frac{\cos^3 L}{R^2}\delta\rho \end{array}\right\}$$

$$-P\rho = -P(R + \delta\rho)$$

The last term, $P\delta\rho$, is to be rejected, as being the product of two small quantities. Now, if we add all vertically, and remark that the first column on the right side represents the terms in an undisturbed orbit, and, therefore, necessarily $= 0$,—

$$0 = +\frac{d^2R}{dt^2}\delta\rho + R\frac{d^2\,\delta\rho}{dt^2} - 2R^2\frac{dV}{dt}\frac{d\,\delta v}{dt} - 2R\left(\frac{dV}{dt}\right)^2\delta\rho - \frac{3}{R}\cos^2 L\sin L\,\delta l$$
$$- \frac{\cos^3 L}{R^2}\delta\rho - P\cdot R$$

Proceeding now to Equation (5).

This equation consists of the single-term $+\frac{d}{dt}\left\{\rho^2\frac{dv}{dt}\right\} - (tf)\cdot\rho = 0$. And here it is to be remarked that, in the ecliptic force transversal to the ecliptic radius, there is no part derived from the earth's central attraction, and the value of $-(tf)\,\rho$ is strictly limited to the small perturbation term $-T\cdot\rho$. And the equation (5) of page 10 becomes $+\frac{d}{dt}\left\{\rho^2\frac{dv}{dt}\right\} - T\cdot\rho = 0$, or $\frac{d}{dt}\left\{(R+\delta\rho)^2\cdot\left(\frac{dV}{dt} + \frac{d\,\delta v}{dt}\right)\right\} - T\cdot\rho = 0$, which, treated in the same manner, gives,—

$$2\left(R\frac{dV}{dt}\right)\cdot\frac{d\,\delta\rho}{dt} + 2\frac{d}{dt}\left(R\frac{dV}{dt}\right)\delta\rho + R^2\frac{d^2\,\delta v}{dt^2} + \frac{d}{dt}(R^2)\frac{d\,\delta v}{dt} - T\cdot\rho = 0$$

Finally, we take Equation (6).

Here it is to be remarked that the force (zf), which is the whole force normal to the ecliptic plane, consists of the sum of — the resolved part of the Earth's and Moon's attraction (which is $-\frac{1}{r^2} \sin 1$ or $-\frac{1}{\rho^2} \cos^3 1 \cdot \sin 1$),—and the disturbing force Z. And, therefore $(zf) = -\frac{1}{\rho^2} \cos^3 1 \cdot \sin 1 + Z$. And the equation (6) of page 10 becomes—

$$+\frac{d^2}{dt^2}(\rho \cdot \tan 1) + \frac{1}{\rho^2} \cdot \cos^3 1 \cdot \sin 1 - Z = 0,$$

$$\text{or } \frac{d^2}{dt^2}\left\{(R+\delta\rho)\cdot(\tan L + \sec^2 L \cdot \delta l)\right\} + \left(\frac{1}{R^2} - \frac{2}{R^3}\delta\rho\right)\cdot \cos^3 1 \cdot \sin 1$$
$$+ \frac{1}{R^2}\left(-2\cos L + 3\cos^3 L\right)\cdot \delta l - Z = 0$$

Or, as in the former instances,—

$$\left\{\begin{array}{lll} +\frac{d^2}{dt^2}(\tan L)\cdot\delta\rho & +2\frac{d}{dt}(\tan L)\cdot\frac{d\,\delta\rho}{dt} & +\tan L\cdot\frac{d^2\,\delta\rho}{dt^2} \\ +\frac{d^2}{dt^2}(R\cdot\sec^2 L)\cdot\delta l & +2\frac{d}{dt}(R\cdot\sec^2 L)\cdot\frac{d\,\delta l}{dt} & +R\cdot\sec^2 L\cdot\frac{d^2\,\delta l}{dt^2} \\ -\frac{2}{R^3}\cos^2 L\cdot\sin L\cdot\delta\rho & +\frac{1}{R^2}(-2\cos L+3\cos^3 L)\cdot\delta l & \\ -Z & & \end{array}\right\} = 0$$

The values which we are seeking, for $\delta\rho$, δv, and δl, as produced by external action, are evidently founded on the values of P, T, and Z. There is a single constant term in P, and all other parts of P, T, Z, are expressed by periodical terms, sines or cosines of various multiples of the time, and these in all algebraical treatment, are absolutely independent. It is clear then that any one of these terms may be treated without writing down any other term; and if we assume that a term of P will be expressed by A · cos mt, and (in connexion with it) that T will be expressed by B · sin mt, and Z by C · cos mt, then the expressions for $\delta\rho$, δv, δl, and every term in the algebraic operations, will depend on cos mt and sin mt and constants connected with them. This consideration introduces great simplicity into all the expressions. For, as the values of R, V, and L, which we have occasion to use, will not depend on mt, or will depend only on terms so far advanced in the series that they never could enter into consideration with those which we do retain, we have no need to use any periodic term in the expansions of the factors of $\delta\rho$, δv, δl, and may confine ourselves to the first terms of the series which expresses each of those factors. Thus,

for each of the following symbols we may substitute 1,

$$R \quad R^2\frac{dV}{dt} \quad R\frac{dV}{dt} \quad R^3\frac{dV}{dt} \quad R\left(\frac{dV}{dt}\right)^2 \quad \cos L \quad R\cdot\sec^2 L \quad \cos L \quad \cos^3 L$$

And we may consider each of the following $= 0$,

$$\frac{d\,R}{dt^2} \quad \sin L \quad \frac{d}{dt}\left(R\cdot\frac{dV}{dt}\right) \quad \frac{d}{dt}(R^2)\tan L \quad \frac{d}{dt}(\tan L) \quad \frac{d^2}{dt^2}(\tan L) \quad \frac{d}{dt}(R\cdot\sec^2 L) \quad \frac{d^2}{dt^2}(R\cdot\sec^2 L)$$
$$\cos^2 L\cdot\sin L$$

The formulæ are given in this shape, for establishing, when desired, the connexion between the points now under consideration and the primary investigations of the theory We must now prepare accurate expressions for the forces to be employed on the new investigations These forces, it is to be remarked, are not modifications of the forces formerly treated, but are the new forces depending on the oblateness of the Earth, not formerly taken into account, and the expressions will be carried here to a greater number of decimals

With these we now proceed The amount of calculations which the investigations have required is great, and it is impossible to exhibit the entire details

As regards the notation employed, it is to be remarked that—

The roman R, T, Z, L, are not further used

The ordinary italics R, T, Z, will be used generally for forces in the directions of, radius projected on the standard plane, direction transversal to radius, in the standard plane, and direction normal to the standard plane

w and nt are indifferently used for Moon's mean longitude

v is used for Moon's true longitude, and λ for Moon's true latitude

r is used for the length of the Moon's radius vector, and ρ for the length of projection of the radius-vector upon the standard plane

Other symbols as in the Tables of Section II (In the last lines of page 144, the letter H has been inadvertently used for u)

The first step is to exhibit the expression of the various powers and combinations of r and ρ The numbers for $\frac{1}{r}$ have been given in Section II, but are repeated here for convenience

Numerical Values of Combinations of Powers of $\frac{1}{r}$ and ρ

All the numbers are to be treated as whole-numbers and to be multiplied by 10^{-4}

Factors for every Series	Subjects of each Series									
	$\frac{1}{r^4}$	$\frac{1}{r}$	$\frac{1}{r^6}$	$\frac{1}{r^7}$	ρ	ρ	ρ^3	$\frac{1}{r}\rho$	$\frac{1}{r^6}\rho$	$\frac{1}{r^7}\rho^3$
Constant	+10095	+10159	+10 30	+10337	+9995	+10006	+10032	+10075	+10049	+10033
Cosine $\lvert l \rvert$	+2201	+2764	+3[illegible]34	+3915	−543	−1084	−1623	+2189	+2147	+2126
Cosine $\lvert 2D-l \rvert$	+4[illegible]5	+513	+676	+820	−96	−188	−276	+434	+427	+425
Cosine $\lvert 2D \rvert$	+367	+175	+591	+716	−75	−145	−210	+368	+372	+366
Cosine $\lvert 2l \rvert$	+210	+301	+465	+532	−15	−15	−1	+210	+211	+[illegible]10
Cosine $\lvert D+l \rvert$	+66	+95	+131	+173	−4	−4	0	+67	+66	+114
Cosine $\lvert 2D-S \rvert$	+24	+9	+35	+41	−5	−11	−15	+23	+2[illegible]	−25
Cosine $\lvert 2D-l-S \rvert$	+19	+4	+39	+34	−4	−8	−12	+18	−12	−15
Cosine $\lvert l-S \rvert$	+13	+15	+15	+21	−3	−6	−9	+12	+12	+11
Cosine $\lvert D \rvert$	−11	−16	−19	−23	+3	+5	+7	−13	−14	−16
Cosine $\lvert l+S \rvert$	−11	−14	−17	−20	+3	+5	+7	−11	−12	−4
Cosine $\lvert 2f-l \rvert$	−5	−10	−12	−14	−1	0	0	−10	−12	−17
Cosine $\lvert 3l \rvert$	+19	+30	+44	+62	−1	0	−3	+20	+19	+16
Cosine $\lvert 4D-l \rvert$	+12	+17	+22	+29	−1	0	+1	+14	+7	−12
Cosine $\lvert S \rvert$	−1	−5	−6	−7	+1	+3	+5	−5	−4	+3
Cosine $\lvert 2D-l+S \rvert$	−5	−6	−7	−9	+1	+2	+3	−6	−4	+3
Cosine $\lvert 2D+S \rvert$	−4	−5	−6	−7	+1	+2	+2	−5	−1	+2
Cosine $\lvert 4D-2l \rvert$	+7	+11	+15	+19	0	0	0	+9	−7	+9
Cosine $\lvert 2D-2l \rvert$	+32	+55	+85	+122	+6	+18	+34	+33	+32	+30
Cosine $\lvert 2D+2l \rvert$	+9	+14	+22	+32	0	0	0	+10	−11	−28
Cosine $\lvert 2D+l-S \rvert$	+1	+5	+6	+7	0	0	0	+4	+2	−4
Cosine $\lvert 4D \rvert$	+6	+8	+11	+15	0	0	0	+6	+10	+5
Cosine $\lvert D+S \rvert$	+2	+2	+3	+3	0	−1	−1	+2	−1	−1
Cosine $\lvert 2D-2f \rvert$	−2	−2	−2	−2	−1	−2	−3	−2	−4	−3
Cosine $\lvert 2l-S \rvert$	+2	+2	+3	+3	−2	−1	+2	+2	+1	−4

Numerical Expansions connected with v.

All the numbers are to be treated as whole-numbers and to be multiplied by 10^{-4}.

For cos v	For sin v	For cos $2v$	For sin $2v$
		+ 5001	
+ 9969 cos $\overline{u}$	+ 9969 sin $\overline{u}$	+ 4936 cos $\overline{2u}$	+ 4937 sin $\overline{2u}$
+ 547 cos $\overline{u+l}$	+ 547 sin $\overline{u+l}$	+ 542 cos $\overline{2u+l}$	+ 544 sin $\overline{2u+l}$
− 551 cos $\overline{u-l}$	− 551 sin $\overline{u-l}$	− 550 cos $\overline{2u-l}$	− 550 sin $\overline{2u-l}$
+ 108 cos $\overline{u+2D-l}$	+ 108 sin $\overline{u+2D-l}$	+ 104 cos $\overline{2u+2D-l}$	+ 104 sin $\overline{2u+2D-l}$
− 114 cos $\overline{u-2D+l}$	− 114 sin $\overline{u-2D+l}$	− 116 cos $\overline{2u-2D+l}$	− 116 sin $\overline{2u-2D+l}$
+ 64 cos $\overline{u+2D}$	+ 64 sin $\overline{u+2D}$	+ 70 cos $\overline{2u+2D}$	+ 70 sin $\overline{2u+2D}$
− 52 cos $\overline{u-2D}$	− 52 sin $\overline{u-2D}$	− 46 cos $\overline{2u-2D}$	− 46 sin $\overline{2u-2D}$
+ 34 cos $\overline{u+2l}$	+ 34 sin $\overline{u+2l}$	+ 49 cos $\overline{2u+2l}$	+ 49 sin $\overline{2u+2l}$
− 4 cos $\overline{u-2l}$	− 4 sin $\overline{u-2l}$	+ 11 cos $\overline{2u-2l}$	+ 11 sin $\overline{2u-2l}$
+ 9 cos $\overline{u+2D+l}$	+ 9 sin $\overline{u+2D+l}$	+ 11 cos $\overline{2u+2D+l}$	+ 1 sin $\overline{2u+2D+l}$
		+ 2 cos $\overline{2u-2D-l}$	+ 2 sin $\overline{2u-2D-l}$
+ 4 cos $\overline{u+2D-S}$	+ 4 sin $\overline{u+2D-S}$	+ 4 cos $\overline{2u+2D-S}$	+ 4 sin $\overline{2u+2D-S}$
− 4 cos $\overline{u-2D+S}$	− 4 sin $\overline{u-2D+S}$	− 4 cos $\overline{2u-2D+S}$	− 4 sin $\overline{2u-2D+S}$
+ 5 cos $\overline{u+2D-l-S}$	+ 5 sin $\overline{u+2D-l-S}$	+ 5 cos $\overline{2u+2D-l-S}$	+ 4 sin $\overline{2u+2D-l-S}$
− 5 cos $\overline{u-2D+l+S}$	− 5 sin $\overline{u-2D+l+S}$	− 5 cos $\overline{2u-2D+l+S}$	− 4 sin $\overline{2u-2D+l+S}$
+ 5 cos $\overline{u+l-S}$	+ 5 sin $\overline{u+l-S}$	+ 6 cos $\overline{2u+l-S}$	+ 4 sin $\overline{2u+l-S}$
− 3 cos $\overline{u-l+S}$	− 3 sin $\overline{u-l+S}$	− 2 cos $\overline{2u-l+S}$	− 2 sin $\overline{2u-l+S}$
− 3 cos $\overline{u+D}$	− 3 sin $\overline{u+D}$	− 3 cos $\overline{2u+D}$	− 2 sin $\overline{2u+D}$
+ 3 cos $\overline{u-D}$	+ 3 sin $\overline{u-D}$	+ 3 cos $\overline{2u-D}$	+ 2 sin $\overline{2u-D}$
− 4 cos $\overline{u+l+S}$	− 4 sin $\overline{u+l+S}$	− 5 cos $\overline{2u+l+S}$	− 4 sin $\overline{2u+l+S}$
+ 2 cos $\overline{u-l-S}$	+ 2 sin $\overline{u-l-S}$	+ 1 cos $\overline{2u-l-S}$	+ 2 sin $\overline{2u-l-S}$
− 16 cos $\overline{u+S}$	− 16 sin $\overline{u+S}$	− 16 cos $\overline{2u+S}$	− 16 sin $\overline{2u+S}$
+ 16 cos $\overline{u-S}$	+ 16 sin $\overline{u-S}$	+ 16 cos $\overline{2u-S}$	+ 16 sin $\overline{2u-S}$
− 11 cos $\overline{u-2D+2l}$	− 11 sin $\overline{u-2D+2l}$	− 17 cos $\overline{2u-2D+2l}$	− 16 sin $\overline{2u-2D+2l}$
− 10 cos $\overline{u+2J}$	− 10 sin $\overline{u+2J}$	− 10 cos $\overline{2u+2J}$	− 10 sin $\overline{2u+2J}$
+ 10 cos $\overline{u-2J}$	+ 10 sin $\overline{u-2J}$	+ 10 cos $\overline{2u-2J}$	+ 10 sin $\overline{2u-2J}$
		+ 3 cos $\overline{2u+3l}$	+ 2 sin $\overline{2u+3l}$
		+ 1 cos $\overline{2u-3l}$	
		+ 2 cos $\overline{2u+2D-l}$	
		+ 2 cos $\overline{2u-4D+l}$	
		− 7 cos $\overline{2u+2D-2l}$	− 6 sin $\overline{2u+2D-2l}$

Numerical Expansions connected with z

All the numbers are to be treated as whole numbers, and are to be multiplied by 10^{-4}

For z				For			
Sine $\overline{f}$	×	+	896	Constant	×	+	40
Sine $\overline{f+l}$	×	+	25	Cosine $\overline{2f}$	×	−	40
Sine $\overline{f-l}$	×	−	73	Cosine $\overline{2f+l}$	×	−	2
Sine $\overline{2D-f}$	×	+	33	Cosine $\overline{2f-l}$	×	+	7
Sine $\overline{2D+f-l}$	×	+	5	Cosine l	×	+	5
Sine $\overline{2D-f-l}$	×	+	12	Cosine $\overline{2D}$	×	−	3
Sine $\overline{2D+f}$	×	+	2	Cosine $\overline{2D-2f}$	×	+	3
Sine $\overline{f+2l}$	×	+	1	Cosine $\overline{2D-2f-l}$	×	+	1
Sine $\overline{2D-f+2l}$	×	+	1	Cosine $\overline{2D-l}$	×	−	1
Sine $\overline{f-2l}$	×	−	1	For z^3			
Sine $\overline{2D-f-S}$	×	+	2				
Sine $\overline{2D-f+S}$	×	−	1	Sine $\overline{f}$	×	+	5
Sine $\overline{2D-f-l-S}$	×	+	1	Sine $\overline{3f}$	×	−	2

The numbers of the last three tables contain all that is necessary, when used in connexion with the external factors, for completing the numerical values of the terms on page 144 Without attempting to exhibit the mass of figures employed in these calculations, I now give only the results for each of the quantities called (B), (C), &c, to (P) It will be remarked that the numbers in the preceding long columns have all been given to the 4th place of decimals, and the factors at the head of the columns which now follow are given to the 10th place of decimals On repeating any of the multiplications, it will be immediately seen that the products, as exhibited below are formed to the 14th place of decimals of unity

Expression of the Force R acting on the Moon in the direction of the projection of Radius Vector on the Plane of the Ecliptic—*completed on next page*

—completed on next page

Each product of numbers is to be treated as a whole-number, and is to be multiplied by 10^{-11}

Heading (B)

Coefficients, to be multiplied by -1388×10^{-10}

+	5056	$\cos\overline{0}$	+	2	$\cos\overline{2u-4D+l}$
+	4975	$\cos\overline{2u}$	−	17	$\cos\overline{2u+S}$
+	1088	$\cos\overline{2u+l}$	+	17	$\cos\overline{2u-S}$
−	3	$\cos\overline{2u-l}$	+	5	$\cos\overline{2u+4D-2l}$
+	208	$\cos\overline{2u+2D-l}$	−	9	$\cos\overline{2u-2D+2l}$
−	8	$\cos\overline{2u-2D+l}$	−	110	$\cos\overline{2u+2f}$
+	184	$\cos\overline{2u+2D}$	+	10	$\cos\overline{2u-2f}$ *
+	22	$\cos\overline{2u-2D}$ *	−	2	$\cos\overline{2u+2f+l}$
+	160	$\cos\overline{2u+2l}$	+	8	$\cos\overline{2u+2l}$
+	4	$\cos\overline{2u-2l}$ *	+	3	$\cos\overline{2u-2D-2l}$
+	50	$\cos\overline{2u+2D+l}$	+	2	$\cos\overline{2u+4D}$
+	3	$\cos\overline{2u-2D-l}$	+	1099	$\cos\overline{l}$
+	5	$\cos\overline{2u+2D-S}$	+	215	$\cos\overline{2D-l}$
−	5	$\cos\overline{2u-2D+S}$	+	186	$\cos\overline{2D}$
+	5	$\cos\overline{2u+2D-l-S}$	+	105	$\cos\overline{2l}$
−	5	$\cos\overline{2u-2D+l+S}$	+	34	$\cos\overline{2D+l}$
+	6	$\cos\overline{2u+l-S}$	+	10	$\cos\overline{3l}$
−	4	$\cos\overline{2u-l+S}$	−	6	$\cos\overline{2f-l}$
−	[illegible]	$\cos\overline{2u+D}$	+	6	$\cos\overline{4D-l}$
+	3	$\cos\overline{2u-D}$	−	3	$\cos\overline{2D-l+S}$
−	7	$\cos\overline{2u+l+S}$	+	17	$\cos\overline{2D-2l}$
+	3	$\cos\overline{2u-l-S}$	−	2	$\cos\overline{2D-2f}$
−	3	$\cos\overline{2u+2f-l}$	+	4	$\cos\overline{4D-2l}$
+	19	$\cos\overline{2u+3l}$	+	6	$\cos\overline{2D+2l}$
+	9	$\cos\overline{2u+4D-l}$	+	2	$\cos\overline{2D-3l}$

Heading (C)

Coefficients, to be multiplied by -3201×10^{-10}

+	446	$\sin\overline{u+f}$	+	16	$\sin\overline{u+2D-l+f}$
−	453	$\sin\overline{u-f}$ *	−	10	$\sin\overline{u-2D+l-f}$
+	98	$\sin\overline{u+l+f}$	−	9	$\sin\overline{u+2D-l-f}$
−	48	$\sin\overline{u-l-f}$	+	17	$\sin\overline{u+2D+f}$
−	50	$\sin\overline{u+l-f}$	−	8	$\sin\overline{u-2D-f}$
+	5	$\sin\overline{u+2D-f}$	+	14	$\sin\overline{u+2l+f}$
−	10	$\sin\overline{u-2D+f}$ *	−	4	$\sin\overline{u-2l-f}$
			+	3	$\sin\overline{u+2D+l+f}$
			−	3	$\sin\overline{u+2l-f}$

Heading (D)

Coefficients, to be multiplied by -4363×10^{-10}

+	10074	$\cos\overline{0}$	+	33	$\cos\overline{2D-2l}$ *
+	2198	$\cos\overline{l}$	−	3	$\cos\overline{2D-2f}$
+	430	$\cos\overline{2D-l}$	+	8	$\cos\overline{4D-2l}$
+	367	$\cos\overline{2D}$	+	11	$\cos\overline{2D+2l}$
+	210	$\cos\overline{2l}$	+	6	$\cos\overline{4D}$
+	67	$\cos\overline{2D+l}$	+	4	$\cos\overline{2D-3l}$
+	19	$\cos\overline{3l}$	+	2	$\cos\overline{4l}$
−	11	$\cos\overline{2f-l}$			
+	12	$\cos\overline{4D-l}$			
+	5	$\cos\overline{2D-l+S}$			

* The terms to which an asterisk is attached are used in further calculations

Expression of the Force R acting on the Moon in the direction of the projection of Radius Vector on the Plane of the Ecliptic—*continued and completed*

Each product of numbers is to be treated as a whole-number, and is to be multiplied by 10^{-11}

Heading (E)

Co efficients, to be multiplied by $+ 3471 \times 10^{-10}$

$+ 5013 \cos 0$	$- 17 \cos \overline{2u+S}$
$+ 4954 \cos \overline{2u}$	$+ 17 \cos \overline{2u-S}$
$+ 1081 \cos \overline{2u+l}$	$+ 4 \cos \overline{2u+4D-2l}$
$- 5 \cos \overline{2u-l}$	$- 10 \cos \overline{2u-2D+2l}$
$+ 207 \cos \overline{2u+2D-l}$	$- 10 \cos \overline{2u+2f}$
$- 9 \cos \overline{2u-2D+l}$	$+ 10 \cos \overline{2u-2f}$ *
$+ 183 \cos \overline{2u+2D}$	$- 2 \cos \overline{2u+2f+l}$
$+ 22 \cos \overline{2u-2D}$ *	$+ 3 \cos \overline{2u+4l}$
$+ 158 \cos \overline{2u+2l}$	$+ 7 \cos \overline{2u+2D+2f}$
$+ 3 \cos \overline{2u-2l}$ *	$+ 5 \cos \overline{2u+3l}$
$+ 49 \cos \overline{2u+2D+l}$	$+ 1093 \cos \overline{l}$
$+ 2 \cos \overline{2u-2D-l}$	$+ 213 \cos \overline{2D-l}$
$+ 5 \cos \overline{2u+2D-S}$	$+ 184 \cos \overline{2D}$
$- 5 \cos \overline{2u-2D+S}$	$+ 104 \cos \overline{2l}$
$+ 5 \cos \overline{2u+2D-l-S}$	$+ 33 \cos \overline{2D+l}$
$- 5 \cos \overline{2u-2D+l+S}$	$+ 10 \cos \overline{3l}$
$+ 8 \cos \overline{2u+l+S}$	$- 9 \cos \overline{2f-l}$
$- 4 \cos \overline{2u-l+S}$	$+ 4 \cos \overline{4D-l}$
$- 3 \cos \overline{2u+D}$	$- 3 \cos \overline{2D-l+S}$
$+ 3 \cos \overline{2u-D}$	$+ 15 \cos \overline{2D-2l}$ *
$- 7 \cos \overline{2u+l+S}$	$- 3 \cos \overline{2D-2f}$
$+ 3 \cos \overline{2u-l-S}$	$+ 3 \cos \overline{4D-2l}$
$- 4 \cos \overline{2u+2f-l}$	$+ 4 \cos \overline{2D+2l}$
$- 2 \cos \overline{2u+2f+l}$	$+ 3 \cos \overline{4l}$
$+ 19 \cos \overline{2u+3l}$	

Heading (F)

Co efficients, to be multiplied by $+ 16003 \times 10^{-10}$

$+ 467 \sin \overline{u+f}$	$- 2 \sin \overline{u-2D+l+f}$
$- 472 \sin \overline{u-f}$ *	$+ 16 \sin \overline{u+2D+f}$
$+ 95 \sin \overline{u+l+f}$	$- 9 \sin \overline{u-2D-f}$
$- 48 \sin \overline{u-l-f}$	$- 5 \sin \overline{u-2l-f}$
$- 49 \sin \overline{u+l-f}$	$+ 2 \sin \overline{u+2D+l-f}$
$+ 4 \sin \overline{u+2D-f}$	$- 2 \sin \overline{u-2D-l+f}$
$- 11 \sin \overline{u-2D+f}$ *	$- 5 \sin \overline{u+2l-f}$
$+ 19 \sin \overline{u+2D-l+f}$	$+ 3 \sin \overline{u+2D+l+f}$
$- 9 \sin \overline{u-2D+l-f}$	
$- 10 \sin \overline{u+2D-l-f}$	

Heading (G)

Co efficients to be multiplied by $+ 18446 \times 10^{-10}$

$+ 40 \cos \overline{0}$	$- 1 \cos \overline{2D-2f-l}$
$+ 9 \cos \overline{l}$	$- 1 \cos \overline{2D+2f-l}$
$+ 2 \cos \overline{2D-l}$	$- 40 \cos \overline{2f}$
$- 1 \cos \overline{D}$	$- 8 \cos \overline{2f+l}$
$+ 1 \cos \overline{2l}$	$- 1 \cos \overline{2D+f}$
$+ 1 \cos \overline{2D-2f}$	$- 4 \cos \overline{l+2f}$

* The terms to which an asterisk is attached are used in further calculations

Expression of the Force T acting on the Moon in the direction in the Plane of the Ecliptic, at right angles to the projection of the Radius Vector—*completed*

Each product of numbers is to be treated as a whole-number, and is to be multiplied by 10^{-11}

Heading (H) Coefficients to be multiplied by $+ 1368 \times 10^{-10}$		*Heading (I)* Coefficients, to be multiplied by $+ 3201 \times 10^{-10}$	
$+ 4974 \sin \overline{2u}$	$- 4 \sin \overline{2u-l+S}$	$- 446 \cos \overline{u+f}$	$+ 10 \cos \overline{u-2D+l-f}$
$+ 1090 \sin \overline{2u+l}$	$- 2 \sin \overline{2u+D}$	$+ 452 \cos \overline{u-f}$ *	$+ 9 \cos \overline{u+2D-l-f}$
$- 3 \sin \overline{2u-l}$	+ [illegible] $\sin \overline{2u-D}$	$- 98 \cos \overline{u+l+f}$	$- 17 \cos \overline{u+2D+f}$
$+ 209 \sin \overline{2u+2D-l}$	$- 6 \sin \overline{2u+l+S}$	$+ 48 \cos \overline{u-l-f}$	$+ 8 \cos \overline{u-2D-f}$
$- 9 \sin \overline{2u-2D+l}$	$+ 4 \sin \overline{2u-l-S}$	$+ 50 \cos \overline{u+l-f}$	$- 14 \cos \overline{u+2l+f}$
$+ 184 \sin \overline{2u+2D}$	$- 4 \sin \overline{2u+2f-l}$	$- 5 \cos \overline{u+2D-f}$	$+ 4 \cos \overline{u-2l-f}$
$+ 22 \sin \overline{2u-2D}$ *	$- 2 \sin \overline{2u-2f+l}$	$+ 10 \cos \overline{u-2D+f}$ *	$- 3 \cos \overline{u+2D+l+f}$
$+ 160 \sin \overline{2u+2l}$	$+ 18 \sin \overline{2u+3l}$	$- 18 \cos \overline{u+2D-l+f}$	$+ 3 \cos \overline{u+2l-f}$
$+ 4 \sin \overline{2u-2l}$ *	$+ 7 \sin \overline{2u+4D-l}$		
$+ 49 \sin \overline{2u+2D+l}$	$- 16 \sin \overline{2u+S}$		
$+ 3 \sin \overline{2u-2D-l}$	$+ 16 \sin \overline{2u-S}$		
$+ 4 \sin \overline{2u+2D-S}$	$+ 5 \sin \overline{2u+4D-2l}$		
$- 4 \sin \overline{2u-2D+S}$	$+ 2 \sin \overline{2u+2D-2l}$		
$+ 4 \sin \overline{2u+2D-l-S}$	$- 8 \sin \overline{2u-2D+2l}$		
− [illegible] $\sin \overline{2u-2D+l+S}$	$- 10 \sin \overline{2u+2f}$		
$+ 6 \sin \overline{2u+l-S}$	$+ 10 \sin \overline{2u-2f}$		
	+ [illegible] $\sin \overline{2u+2D+2l}$ *		
	$+ 3 \sin \overline{2u-2D-2l}$		
	$+ 2 \sin \overline{2u+4D}$		

* The terms to which an asterisk is attached are used in further calculations

Expression of the Force Z acting on the Moon in the direction normal to the Plane of the Ecliptic—*completed on next page*

Each product of numbers is to be treated as a whole-number, and is to be multiplied by 10^{-11}

Heading (J)

Co efficients, to be multiplied by -3201×10^{-10}

+	10043	cos	$\overline{u}$ *	+	10	cos	$\overline{u+4D-l}$
+	1844	cos	$\overline{u+l}$	+	4	cos	$\overline{u-4D+l}$
+	548	cos	$\overline{u-l}$	−	16	cos	$\overline{u+S}$
+	321	cos	$\overline{u+2D-l}$	+	16	cos	$\overline{u-S}$
+	101	cos	$\overline{u-2D+l}$	+	6	cos	$\overline{u+4D-2l}$
+	270	cos	$\overline{u+2D}$	+	2	cos	$\overline{u-4D+2l}$
+	109	cos	$\overline{u-2D}$	+	16	cos	$\overline{u+2D-2l}$
+	198	cos	$\overline{u+2l}$	+	15	cos	$\overline{u-2D+2l}$
+	42	cos	$\overline{u-2l}$ *	+	2	cos	$\overline{u+2D-3l}$
+	61	cos	$\overline{u+2D+l}$	+	2	cos	$\overline{u-2D+3l}$
+	15	cos	$\overline{u-2D-l}$	−	10	cos	$\overline{u+2f}$
+	5	cos	$\overline{u+2D-S}$	+	10	cos	$\overline{u-2f}$
−	5	cos	$\overline{u-2D+S}$	+	10	cos	$\overline{u+2D+2l}$
+	5	cos	$\overline{u+2D-l-S}$	+	2	cos	$\overline{u-2D-2l}$
−	5	cos	$\overline{u-2D+l+S}$	+	2	cos	$\overline{u+4l}$
+	7	cos	$\overline{u+l-S}$	−	2	cos	$\overline{u+2D-l+S}$
−	5	cos	$\overline{u-l+S}$	−	2	cos	$\overline{u-2D+l-S}$
−	3	cos	$\overline{u+D}$	+	4	cos	$\overline{u+4D}$
+	3	cos	$\overline{u-D}$	+	2	cos	$\overline{u-4D}$
−	6	cos	$\overline{u+l+S}$				
+	4	cos	$\overline{u-l-S}$				
−	6	cos	$\overline{u+2f-l}$				
−	4	cos	$\overline{u-2f+l}$				
+	20	cos	$\overline{u+3l}$				
+	4	cos	$\overline{u-3l}$				

Heading (K) and (I)

Co efficients to be multiplied by -7378 for (K), *Co efficients* to be multiplied by -4383 for (I), each $\times 10^{-10}$

+	901	sin	$\overline{f}$ *	+	24	sin	$\overline{2D+f}$
+	146	sin	$\overline{l+f}$	+	17	sin	$\overline{2l+f}$
−	50	sin	$\overline{l-f}$	+	3	sin	$\overline{2D+l-f}$
+	16	sin	$\overline{2D-f}$	−	2	sin	$\overline{2l-f}$
+	29	sin	$\overline{2D-l+f}$	+	2	sin	$\overline{3l+f}$
−	9	sin	$\overline{2D-l-f}$	+	5	sin	$\overline{2D+l+f}$

Heading (M)

Co efficients to be multiplied by $+3471 \times 10^{-10}$

+	220	sin	$\overline{2u+f}$	+	3	sin	$\overline{2u+2D+l+f}$
−	224	sin	$\overline{2u-f}$ *	+	2	sin	$\overline{2u-2D-l+f}$
+	60	sin	$\overline{2u+l+f}$	+	449	sin	$\overline{f}$ *
−	9	sin	$\overline{2u-l-f}$	+	74	sin	$\overline{l+f}$
−	36	sin	$\overline{2u+l-f}$	+	73	sin	$\overline{l-f}$
−	13	sin	$\overline{2u-l+f}$	+	8	sin	$\overline{2D-f}$
−	7	sin	$\overline{2u-2D+f}$	+	14	sin	$\overline{2D-l+f}$
+	11	sin	$\overline{2u+2D-l+f}$	−	3	sin	$\overline{2D-l-f}$
−	2	sin	$\overline{2u-2D+l-f}$	+	13	sin	$\overline{2D+f}$
−	9	sin	$\overline{2u+2D-l-f}$	+	10	sin	$\overline{2l+f}$
−	3	sin	$\overline{2u-2D+l+f}$	+	2	sin	$\overline{2D+l-f}$
+	10	sin	$\overline{2u+2D+f}$	+	9	sin	$\overline{2l-f}$ *
−	3	sin	$\overline{2u-2D-f}$				
+	9	sin	$\overline{2u+2l+f}$				
−	3	sin	$\overline{2u+2l-f}$				

* The terms to which an asterisk is attached are used in further calculations

Expression of the Force Z acting on the Moon in the direction normal to the Plane of the Ecliptic—*completed*

Each product of numbers is to be treated as a whole number, and is to be multiplied by 10^{-15}

Heading (N) *Co efficients*, to be multiplied by $+ 16003 \times 10^{-10}$	*Heading* (P) *Co efficients* to be multiplied by $+ 18446 \times 10^{-10}$
$+ \; 40 \cos \overline{u}$ *	$+ \; 6 \sin \overline{f}$
$+ \; 6 \cos \overline{u+l}$	$- \; 2 \sin \overline{3f}$
$+ \; 6 \cos \overline{u-l}$	
$+ \; 1 \cos \overline{u+2D-l}$	
$+ \; 1 \cos \overline{u-2D+l}$	
$+ \; 1 \cos \overline{u-l+2f}$	
$- \; 1 \cos \overline{u+l-2f}$	
$- \; 5 \cos \overline{u+l+2f}$	
$- \; 5 \cos \overline{u-l-2f}$	
$- \; 20 \cos \overline{u+2f}$	
$- \; 20 \cos \overline{u-2f}$ *	
$+ \; 1 \cos \overline{u+2D-2f}$	
$+ \; 1 \cos \overline{u-2D+2f}$	

* The terms to which an asterisk is attached are used in further calculations

RELATIONS BETWEEN THE VERY SMALL FORCES OF LONG PERIOD IN THE PLANE PARALLEL TO THE ECLIPTIC, AND THE PERTURBATIONS WHICH THEY PRODUCE IN THE MOON'S PLACE

We shall now consider the effect which the disturbing force will produce on the value of [illegible] and v (the ecliptic radius vector and the longitude of the Moon) omitting for the present the latitude, and the small effects of the latitude on the perturbation of ρ and [illegible] We shall suppose R and T expressed (by expansion of the quantities lately formed before introduction of the terms of oblateness) in cosines and sines of multiples of the time, mt being used as the (general) argument of corresponding terms in R and T and R cos mt and T sin mt being the corresponding parts of R and T, applying to any one of those terms And without referring to the origin of the perturbation terms we shall consider the Moon's orbit as fundamentally a circle (neglecting all minor effects of eccentricity and other elements [illegible] to the special inquiry), the radius of the circle being $= 1$ the same unit of [illegible] applying [illegible] the curve of the circle (thus instead of $1''$ we may use $\cdot 000000$ [illegible]) the central [illegible] and the undisturbed velocity of the Moon in longitude or n, $= 1$ We shall consider each new inequality of the Moon's motion, which is superposed on that fundamental circular motion as depending on the same argument mt on which the perturbing causes depend, and we shall form the equation by assuming algebraical expressions for the Moon's displacements in ρ and v and finding what must be the numerical values of the external force R' cos mt and T sin mt which will produce them And thence, by ordinary reversion of equations we shall infer the numerical values of the perturbations of ρ and v, which will be produced by given values of R and T

Now if the disturbing force, whose effects we are considering, did not exist the elements of orbital motion would be ρ (for radius vector), and v or nt (the angle made by [illegible] with the axis of y) The ordinates of a point on that undisturbed orbit would be, [illegible] or $x = \rho$ sin nt, $y = \rho$ cos nt, and, considering $\rho = 1$, and the Earth's attractive force at that distance $= 1$, the force in x would be sin nt, and that in y would be cos nt The disturbing force now under consideration may be represented (for the moment only) by F making with the axis of y the angle mt, and therefore making with ρ the angle $|\overline{mt - nt}|$, and therefore, if not modified by other considerations, producing in ρ a force F cos $|\overline{mt - nt}|$ and in [illegible] a force F sin $|\overline{mt - nt}|$.

Judging from analogies of other lunar terms, it appears not improbable that a disturbance of the length of the radius vector may be expressed by cosine of the periodical argument [illegible] (on which the immediate cause of perturbation depends), and that a disturbance transverse to the radius vector may depend, possibly with a different coefficient, on the sine of the same argument We shall use the letters p and q for these two co-efficients The investigation then takes the following form —

Using x, y, ρ, and v, for the undisturbed place of the Moon and x', y', ρ', and v' for the place as affected by the new force,

$$x = \rho \sin v,\ y = \rho \cos v, \quad x' = \rho' \sin v',\ y' = \rho' \cos v',$$
$$\rho' = \rho + p \cos |\overline{mt - nt}|, \quad v' = v + q \sin |\overline{mt - nt}|$$

$$\sin v' = \sin v + q \cdot \cos v \cdot \sin \overline{mt - nt} = \sin \overline{nt} + q \cdot \cos \overline{nt} \cdot \sin \overline{mt - nt},$$

$$\cos v' = \cos v - q \cdot \sin v \cdot \sin \overline{mt - nt} = \cos \overline{nt} - q \cdot \sin \overline{nt} \cdot \sin \overline{mt - nt}.$$

These and the following investigations are limited to the first power of p and q and of their resultant terms.

ρ will be considered $= 1$.

$$x = \sin \overline{nt} + \tfrac{p}{2}\left(\sin \overline{mt} - \sin \overline{mt - 2nt}\right) + \tfrac{q}{2}\left(\sin \overline{mt} + \sin \overline{mt - 2nt}\right),$$

$$= \text{sine}\, \overline{nt} + \left(\tfrac{p}{2} + \tfrac{q}{2}\right) \text{sine}\, \overline{mt} - \left(\tfrac{p}{2} - \tfrac{q}{2}\right) \text{sine}\, \overline{mt - 2nt},$$

$$\frac{d^2x}{dt^2} = \begin{cases} - n^2 \cdot \sin \overline{nt} \\ - \left(\tfrac{p}{2} + \tfrac{q}{2}\right) \cdot m^2 \cdot \sin \overline{mt} \\ + \left(\tfrac{p}{2} - \tfrac{q}{2}\right) \cdot (m - 2n)^2 \cdot \sin \overline{mt - 2nt} \end{cases}$$

$$y = \cos \overline{nt} + \tfrac{p}{2}\left(\cos \overline{mt} + \cos \overline{mt - 2nt}\right) + \tfrac{q}{2}\left(\cos \overline{mt} - \cos \overline{mt - 2nt}\right),$$

$$= \cos \overline{nt} + \left(\tfrac{p}{2} + \tfrac{q}{2}\right) \cdot \cos \overline{mt} + \left(\tfrac{p}{2} - \tfrac{q}{2}\right) \cos \overline{mt - 2nt}.$$

$$\frac{d^2y}{dt^2} = \begin{cases} - n^2 \cdot \cos \overline{nt} \\ - \left(\tfrac{p}{2} + \tfrac{q}{2}\right) \cdot m^2 \cdot \cos \overline{mt} \\ - \left(\tfrac{p}{2} - \tfrac{q}{2}\right) \cdot (m - 2n)^2 \cdot \cos \overline{mt - 2nt} \end{cases}$$

These expressions represent the entire forces which are acting on the Moon in the directions x and y respectively.

The force arising from their combination, acting in the direction parallel to the undisturbed radius ρ (which makes with y the angle nt), will be,—

$$+ \sin \overline{nt} \cdot \frac{d^2x}{dt^2} + \cos \overline{nt} \cdot \frac{d^2y}{dt^2},$$

$$\text{or, } - n^2 - \left(\tfrac{p}{2} + \tfrac{q}{2}\right) \cdot m^2 \cdot \cos \overline{mt - nt} - \left(\tfrac{p}{2} - \tfrac{q}{2}\right) \cdot (m - 2n)^2 \cdot \cos \overline{mt - nt},$$

$$\text{or, } - n^2 + \{p \cdot (- m^2 + 2mn - 2n^2) + q \cdot (- 2mn + 2n^2)\} \times \cos \overline{mt - nt},$$

and this may be received also as the expression for the entire force along the actual radius vector, complete to the first order of small quantities. But a part of this force is the gravitational attraction towards the Earth, or $-\left(\frac{n^2}{\rho'^2}\right)$, which is equal to $\frac{-n^2}{(1 + p \cdot \cos \overline{mt - nt})^2}$ $= - n^2 + 2n^2p \cdot \cos \overline{mt - nt}$. Subtracting this from the expression for the entire force, there remains for the disturbing force in direction of ecliptic radius-vector,—

$$\{p\,(- m^2 + 2mn - 4n^2) + q\,(- 2mn + 2n^2)\} \times \cos \overline{mt - nt},$$

and this is the complete expression for R, or $R' \cos \overline{mt - nt}$, the radial force which is required in order to maintain the supposed motion.

The force which is acting at right angles to the ecliptic radius-vector is,—

$$\cos \overline{|\, nt \,|}\ \frac{d^2 x}{dt^2} - \sin \overline{|\, nt \,|}\ \frac{d^2 y}{dt^2},$$

tending to accelerate the Moon's motion This is found to be,—

$$-\left(\frac{p}{2} + \frac{q}{2}\right) m^2 \ \sin \overline{|\, mt - nt \,|} + \left(\frac{p}{2} - \frac{q}{2}\right)(m - 2n)^2 \ \sin \overline{|\, mt - nt \,|}\,,$$

$$\text{or, } \left\{ p\,(-\,2mn + 2n^2) + q\,(-\,m^2 + 2mn - 2n^2) \right\} \times \sin \overline{|\, mt - nt \,|}$$

But the force in the undisturbed radius-vector, which we have deduced from the expressions above, is not the force in the true radius-vector which connects the disturbed Moon with the Earth's centre, but is the force defined by the formula $\sin \overline{|\, nt \,|}\ \frac{d^2 x}{dt^2} + \cos \overline{|\, nt \,|}\ \frac{d^2 y}{dt^2}$, and acts, therefore, in the line defined by $\sin \overline{|\, nt \,|}$ and $\cos \overline{|\, nt \,|}$ that is, it acts in the line parallel to the radius drawn from the Earth's centre to the point which is distant from the Moon, in the direction transversal to the radius vector, by $-\, q \ \sin \overline{|\, mt - nt \,|}$ And this introduces, into the expression for disturbance of longitude, the error $-\, q \ \sin \overline{|\, mt - nt \,|}$, and this error must be corrected by introducing into T the addition $+\, q \ \sin \overline{|\, mt - nt \,|}$ or $+\, qn^2 \ \sin \overline{|\, mt - nt \,|}$, ($n^2$ being $= 1$) The force tending to accelerate the Moon's motion is, therefore,—

$$\left\{ p\,(-\,2mn + 2n^2) + q\,(-m^2 + 2mn - n^2) \right\} \times \sin \overline{|\, mt - nt \,|}\,,$$

and this is the complete expression for T or T' $\sin \overline{|\, mt - nt \,|}$, the transversal force which is required to maintain the assumed motion

These expressions for R and T apply equally well, both symbolically and numerically, when the sign of m is negative

Adopting for the future the value $n = 1$, we have now the two equations,—

$$R' = p \ \left\{ -\,3 - (1 - m)^2 \right\} + q \ \left\{ 2 - 2m \right\},$$

$$T' = p \ \left\{ 2 - 2m \right\} - q \ \left\{ (1 - m)^2 \right\},$$

from which, p and q are to be determined in multiples of R' and T' by the ordinary process for two simple equations containing two unknown quantities In the preceding operations, p and R' are numerical quantities, which in applications, multiply cosines, and q and T' are numerical quantities, which, in applications, multiply sines For a number of values of $\frac{n}{m}$, abundantly sufficient for further applications, I have computed and solved, separately, the equations of which the results are given in the following Table It will be remarked that $\frac{n}{m}$ is the same as "the number of geometrical lunations in which the disturbing force, under consideration, goes "through its period" It is also to be noticed that the assumption, that the angular motion mt is positive, implies that the apparent revolution of the Moon's true centre round its unmodified central place is in the same direction as the Moon's revolution round the Earth, if these directions are opposed, m is negative, and the roots of the equations are slightly altered

Computations of First and Second Tables of numerical values of p and q for given numerical values of R' and T'

First Table, for computing p and q when m is positive					Second Table, for computing p and q when m is negative				
Computation of Values of p and q by sums of Multiples of given Values of R' and T'					Computation of Values of p and q by sums of Multiples of given Values of R' and T'				
Values of $\frac{n}{m}$	Computation of p		Computation of q		Values of $\frac{n}{m}$	Computation of p		Computation of q	
	Factors of R'	Factors of T'	Factors of R'	Factors of T'		Factors of R'	Factors of T'	Factors of R'	Factors of T'
+ 1	− 0 33	0 00	0 00	0 00	− 1	− 0 33	− 0 33	− 0 33	− 0 58
+ 2	+ 1 33	+ 5 33	+ 5 33	+ 17 33	− 2	− 0 80	− 1 07	− 1 07	− 1 87
+ 3	+ 1 80	+ 5 40	+ 5 40	+ 13 95	− 3	− 0 29	− 1 93	− 1 93	− 3 46
+ 4	+ 2 29	+ 6 00	+ 6 10	+ 14 48	− 4	− 1 18	− 2 64	− 2 84	− 5 19
+ 5	+ 2 78	+ 6 94	+ 6 94	+ 15 80	− 5	− 2 27	− 3 79	− 3 79	− 7 01
+ 6	+ 3 27	+ 7 85	+ 7 85	+ 17 41	− 6	− 2 77	− 4 75	− 4 75	− 8 87
+ 7	+ 3 77	+ 8 79	+ 8 80	+ 19 16	− 7	− 3 26	− 5 72	− 5 72	− 10 76
+ 8	+ 4 -7	+ 9 75	+ 9 75	+ 20 99	− 8	− 3 77	− 6 70	− 6 70	− 12 62
+ 9	+ 4 77	+ 10 72	+ 10 72	+ 22 85					
+ 10	+ 5 26	+ 11 70	+ 11 70	+ 24 76					
+ 20	+ 10 26	+ 21 59	+ -1 59	+ 44 35					
+ 30	+ 15 25	+ 31 56	+ 31 56	+ 64 23					
+ 40	+ 20 25	+ 41 54	+ 41 55	+ 84 17					
+ 50	+ 25 25	+ 51 54	+ 51 54	+ 104 13					
+ 60	+ 30 25	+ 61 53	+ 61 53	+ 124 11	− 60	− 29 75	− 58 53	− 59 53	− 116 12
+ 70	+ 35 25	+ 71 52	+ 71 52	+ 144 09					
+ 80	+ 40 25	+ 81 52	+ 81 52	+ 164 08					
+ 90	+ 45 25	+ 91 52	+ 91 52	+ 184 07					
+ 100	+ 50 25	+ 101 52	+ 101 52	+ 204 07					
+ 125	+ 62 72	+ 126 52	+ 126 52	+ 254 05	− 125	− 62 40	− 125 52	− 125 5-	− 246 32
+ 250	+ 125 25	+ 251 50	+ 251 50	+ 504 02	− 250	− 124 7-	− 248 56	− 248 56	− 496 36

Our selection of the forms $p \cos \overline{|mt - nt|}$ and $q \sin \overline{|mt - nt|}$ was based upon knowledge that the terms to be supplied for neutralising the recognised inequalities (resulting from the assumption, in the ordinary Lunar Theory, of spherical form of the Earth, and exhibited in the Table of Forces above), must generally be such as would produce *cosines* for terms in the radial direction, and *sines* for terms in the Moon's movement parallel to the ecliptic. There are, however, several inequalities (of which two will engage further attention) which require that the radial term be a sine, and the term of ecliptic parallel be a cosine. For these a separate investigation is required. For distinction, among terms generally similar, we will use the capital letters M, P, Q, in those places where m, p, q have been used in the late investigation.

Let $\rho' = 1 + P \sin \overline{|Mt - nt|}$, $v' = nt + Q \cos \overline{|Mt - nt|}$

Then, taking every step in the same order as in the last investigation,—

New force required in the radial direction =

$$P \times \left\{ - M^2 + 2Mn - 4n^2 \right\} \sin \overline{|Mt - nt|} + Q \times \left\{ + 2Mn - 2n^2 \right\} \times \sin \overline{|Mt - nt|}$$

New force required in the transversal direction parallel to the ecliptic =

$$P \times \left\{ + 2Mn - 2n^2 \right\} \times \cos \overline{|Mt - nt|} + Q \times \left\{ - M^2 + 2Mn - 2n^2 \right\} \times \cos \overline{|Mt - nt|}$$

Putting R'' and T'' for the factors of the trigonometrical terms, and 1 for n, we must have,—

$$R'' = P \times \left\{ - M^2 + 2M - 4 \right\} + Q \times \left\{ + 2M - 2 \right\}$$

$$T'' = P \times \left\{ + 2M - 2 \right\} \qquad + Q \times \left\{ - M^2 + 2M - 1 \right\}$$

And, it being understood that R'' and T'' will be given in subsequent Tables, we are to infer from these equations the values of P and Q

To ascertain clearly the nature of the results which will be given by these formulæ, I have solved the equations, and have calculated in detail, in the same manner as for the Tables lately exhibited, the numerical values of P and Q for the values of $\frac{n}{m} = 2{\cdot}50$ The result is,—

Third Table for computing p and q				
Computation of Values of P and Q from given Values of R'' in sines and T'' in cosines, applying to one value of $\frac{n}{m}$				
Value of $\frac{n}{m}$	Computation of P		Computation of Q	
	Factor of R''	Factor of T''	Factor of R'	Factor of T''
$+ 2{\cdot}50$	$- 1{\cdot}26$	$- 2{\cdot}51$	$+ 2{\cdot}51$	$- 5{\cdot}01$

Thus it appears that the factors for forming the values of P and Q are the same numerically as those for the values of p and q, but the signs are changed in the first and fourth columns There would be no advantage in further extention of these calculations

Here P and R'' are factors of sines, and Q and T'' are factors of cosines

We can now proceed with the actual calculations for selected terms

Our ultimate object is to determine the values of p and q where they are so large as to be observable with astronomical instruments The collection of arguments {symbolical multiples of t with their co-efficients} from which we must make a selection, are those of the "*Expression* of forces R" {including (*B*), (*C*), (*D*), (*E*), (*F*), (*G*), each affected by its proper heading} for the force R, in the direction of the Moon's radius vector, and the "*Expression* of forces T" {including the terms (*H*), (*I*)} for the force T, transversal to the radius-vector Both these collections are to be used in the First, Second, and Third Tables, for determining the magnitudes of p and q The magnitude of each co efficient is important, but another multiplier, equally important, is that given by the multipliers in the three tables, which vary through a great range of magnitude

Inspection of the three tables will show immediately that, after the few first lines, the magnitude of the multiplier of R' or of T' is closely proportional to that of $\frac{n}{m}$, or is inversely proportional to m or to the co efficient of t in the argument of the perturbation terms Bearing

these points in mind, I have selected for trial the six following terms The "movements" or multiples of t in each term are the factors of the movement of the Moon in longitude, and if, as in other places, we consider the velocity of the undisturbed Moon as $= 1$, the "movements" in each term are the numerical measures of the Moon's movement in her orbit To secure, therefore, with greatest probability, the terms of perturbation which will be most conspicuous, we must bring under our investigation those periodical terms in whose arguments the co-efficient of t is small

Elements of the selected Arguments

Order of Term and Argument	Movement of each part in Longitude	Movement of complete Argument in Longitude	Value of $\frac{n}{m}$
No 4 $\vert \overline{2u - 2D} \vert$	+ 2 0000000 − 1 8503974	+ 0 1496026	+ 6 68
No 5 $\vert \overline{2u - 2l} \vert$	+ 2 0000000 − 1 9830960	+ 0 0169040	+ 59 13
No 19 $\vert \overline{2D - 2l} \vert$	+ 1 8503974 − 1 9830960	− 0 1326986	− [illegible] 54
No 51 $\vert \overline{2u - 2f} \vert$	+ 2 0000000 − 2 0080438	− 0 0080438	−124 32
No 301 $\vert \overline{u - f} \vert$	+ 1 0000000 − 1 0040219	− 0 0040219	− [illegible] 64
No 304 $\vert \overline{u - 2D + f} \vert$	+ 1 0000000 − 0 8463755	+ 0 1536245	+ 6 51

We may now select, from the Tables which precede the last, the multipliers required for deducing p and q from R' and T' for each argument These we shall call the "Orbital Factors"

Argument	For Computation of p		For Computation of q	
	Orbital Factor of R'	Orbital Factor of T'	Orbital Factor of R'	Orbital Factor of T'
$\vert \overline{2u - 2D} \vert$	+ 3 61	+ 8 49	+ 8 49	+ 18 60
$\vert \overline{2u - 2l} \vert$	+ 29 82	+ 60 66	+ 60 66	+ 122 37
$\vert \overline{2D - 2l} \vert$	− 3 50	− 6 24	− 6 24	− 11 75
$\vert \overline{2u - 2f} \vert$	− 62 40	− 123 02	− 123 02	− 248 0
$\vert \overline{u - f} \vert$	+ 124 6	− 248 51	− 248 51	+ 496 4
$\vert \overline{u - 2D + f} \vert$	− 3 72	+ 8 35	+ 8 35	− 18 30

Final Comp tations f p and q

Trigonometrical T rm	[... 2D]	[... l]
Column ( ) *Heading* / *Co-efficient* } (Product ( )	(B) 388 } 305 o–	(B) 388 / 4 } – 56
Colum ( ) *Heading* / *Co-efficie* } Produc ( )	(H) 147 o– } 764	(H) 147 / 3 } 64
Column ( ) *Heading* / *Co-efficient* } Product (3		
Sum Preparation of E' (4)	459	48
Orbital Factor of E' for p	1 6	29 8
Product = First Pa of *Preparation for p* (5)	657	43
Column ( ) *Heading* / *Co-efficient* } Product (6)	(H) 388 –4 } 305	(H) 388 / 4 } 56
Orbital Facto of T^2 for p	8 49	60 66
Product = Second Part of *Preparation for p* (7)	2589	3397
(5) (7). Total *Preparation for p* (8)	4246	4808
$\frac{a^2}{4843}$ and Product, for Complete Expression of p in arcsegesimal seconds }	06265 o⁵ / o″ 000l759	06 65 o⁵ / 000958
	sin. [... 2D]	sin [... l]
Preparation f E' from (4) / Orbital Factor of R for q } Product (9)	459 / 8 49 } 3897 o–	48 / 60 66 } 29
Column ( ) *Heading* and *Co-efficient.* / Orbital Factor of T' for q } Product ( )	(H) 388 / o– / 8 60 } 5479	(H) 388 / 4 / 37 } 6684
Sum Total Preparation fo q ( )	9376	9605
$\frac{a^2}{4843}$ sin.	06 65 o⁵	06265 o⁵
Product = Complete Expression of q in arcsegesimal seconds }	= o′ 00 934	= o″ 00 9805
	cos. [... 2D]	cos. [... l]

Γ 1C] tat f p 1 q

⌈D l⌉	⌈ f⌉	⌈ f⌉	⌈− D f⌉
) 388 } 36 7) −4383 } 446 33) 34 } 5 5 6 3 5 4 64	(H) 388 } 39 (I) 347 } 347 8 6 4 979	(C − 3 } 4469 45 (Γ) 6 3 }− 75584 47 6 65 − 4 6 76 87	(C) 3 } 3 (Γ) 6 3 } 76 − 14 3 7 5357
− 4	(H) 388 } 39 3 7 97	(Γ) 3 }− 4469 45 48 5 3595547	(Γ) 3 } 3 × 8 35 67
4 64 6 65 838 ⌈D l⌉	4 8 6 65 9 9 1 ⌈ √⌉	4 47 6 65 3 43 ⌈ f⌉	8 9 656 ⌈− D f⌉
− 6 } 7 44 − 6 4 7 44 6 65 735 ⌈ I l⌉	8 } 5584 3 (H) 388 } 81 48 54 6 6 65 395 ⌈ √⌉	−6 65 } − 5 71653 48 5 (Γ) 3 } 7 8 733 45 496 4 −86996386 6 65 − 7 91457 ⌈ −f⌉	44 } 4 8 35 (Γ) 3 } − 8 3 −58578 6 65 8 ⌈− D f⌉

The results now exhibited are those which we might properly expect from combination of the given values in the Tables of "Expressions of the Forces in the Radial and Transversal Directions derived from the Tabular Elements of the Earth's Action," with the "Co efficients formed purely from the Elements of the Earth's Oblateness"

I am disconcerted by the discordance between my results and those which have been obtained by other theorists But, after careful revision of my work, I can make no change in my numbers The numerical parts are not difficult of verification The principal theoretical point is the form of connexion between "the assumed established forces in direct and transversal measures in regard to radius vector," on the one hand, and "the resulting disturbances of the Moon's place as referred to the same directions," on the other hand At present, I will only remark on this, that the two forces in the direct and transversal measures are of the same order, and that I therefore deem it indispensable that both be included in one comprehensive treatment

I shall, as opportunity serves, endeavour to re-verify the whole process

PERTURBATIONS OF THE MOON IN THE DIRECTION NORMAL TO THE PLANE OF THE ECLIPTIC

The measures of the forces in the boxes of (J), (K), (L), (M), (N), (R), are in the direction normal to the plane of xy, directed from that plane, and we may treat those movements without any reference to the forces or changes of forces in the preceding terms (B) to (I), or any other forces, except they are extremely large, which is not the case here The radius-vector only is a large term, and the Earth's attraction is large, and their whole effect must not be omitted We shall consider the Moon's orbit, before the new inequalities are introduced, as a circle, whose radius is 1, in the plane of xy and shall suppose the Moon to travel with the speed 1 in her orbit

Now let a force $+ Z$ in the direction $+ z$ act upon the Moon The Moon's ordinate, at the time t, would, if there were no other force, become $\int_t \int_t Z$ But the existence of the displacement Z will introduce another force The attraction of the central body will now be in a direction inclined to the plane xy, and will tend to diminish the effect of the force first considered Let z be the true elevation of the Moon above the plane xy Then the force produced by the inclined attraction, tending to raise the Moon from the plane, will be $- \frac{z}{\text{radius vector}} = - z$ And the whole force acting on the Moon in the direction of z will be $Z - z$ And the equation for the Moon's motion will be,—

$$\frac{dz}{dt} + z = Z,$$

of which the integral is,—

$$z = \cos t \int_t \frac{1}{\cos t} \int_t Z \cos t$$

Without using this general formula, it is easily seen that, for any term of Z, of the form $m \left\{ \begin{matrix} \text{sine} \\ \text{cosine} \end{matrix} \right\} nt$, the term of the ordinate will be $\frac{m}{1 - n} \times$ term of Z When Z is a constant, or a multiple of t, the value of z is the same as that of Z

There are two circumstances upon which our selection of terms from the *Expressions* of the force Z will depend The first is the magnitude of the external numerical multiplier m The second is the smallness of the divisor $1 - n^2$, or the near approach of n to 1 For the first of these I propose to take,—

$$\text{The first and largest term of } (J) = - 3201 \times 10^{-10} \times + 10043 \times 10^{-4} \times \cos u,$$
$$\text{and that of } (N) = + 16003 \times 10^{-10} \times + 40 \times 10^{-4} \times \cos u$$

For the others I fix upon,—

$$\begin{array}{lllll}
\text{In } (J) & - 3201 \times 10^{-10} \times & + 42 \times 10^{-4} & \times \cos & \overline{\lvert u - 2l \rvert}, \\
\text{In } (K) \text{ and } (L) & - 11761 \times 10^{-10} \times & + 901 \times 10^{-4} & \times \sin & \lvert f \rvert, \\
\text{In } (M) & + 3471 \times 10^{-10} \times & - 224 \times 10^{-4} & \times \sin & \overline{\lvert 2u - f \rvert}, \\
\text{In } (M) & + 3471 \times 10^{-10} \times & + 449 \times 10^{-4} & \times \sin & \lvert f \rvert, \\
\text{In } (N) & + 16003 \times 10^{-10} \times & - 20 \times 10^{-4} & \times \cos & \overline{\lvert u - 2f \rvert}
\end{array}$$

The first and largest terms produce $-0''.0656 \cos u$ This result, whatever were its magnitude, would be useless, as merely altering the place of the node by a constant quantity

For the other terms, each of which is to be multiplied by the factor of variation $\frac{1}{1 - n^2}$, we have the following elements —

	For $\overline{\lvert u - 2D \rvert}$	For f	For $\overline{\lvert 2u - f \rvert}$	For $\lvert u - 2f \rvert$
n	-0.983096	$+1.004022$	$+0.995976$	-1.008064
$\frac{1}{1-n}$	$+29.829$	-124.08	$+124.56$	-61.91

The co efficients, as first formed in terms of the Radius of the Moon's Orbit, are to be multiplied by $\frac{10^{-5}}{4348}$ for conversion into sexagesimal seconds, and finally we obtain the following expressions for disturbances produced by the Earth's oblateness acting on the Moon in the normal to the plane of the Ecliptic —

$$\begin{array}{rl}
 & '' \\
- 0.065 & \cos u \\
+ 0.008 & \cos \overline{\lvert u - 2l \rvert} \\
- 22.730 & \sin f \\
+ 0.191 & \sin \overline{\lvert 2u - f \rvert} \\
- 0.041 & \cos \overline{\lvert u - f \rvert}
\end{array}$$

NUMERICAL LUNAR THEORY

Section XII.—Effect of a Change in the Position of the Plane of the Solar Ecliptic, on the Apparent Place of the Moon

The planets of the Solar System are undoubtedly, at all times, reciprocally disturbing (in a very minute degree) the fundamental elements of their motion round the Sun. It is not probable that any of their effects, except those which are constantly repeated in the same direction, will be sufficiently large to be remarked by terrestrial observers. One of these, however, is a progressive disturbance in the position of the plane of a planet's orbit round the Sun, causing that plane, in every successive revolution of the planet, to be more and more inclined to its ancient position, turning continually (though slowly) round an axis which passes through the centre of the Sun.

In the combined effect of all the attractions of these bodies, all the bodies are undoubtedly disturbed. But we shall here confine our attention to the Sun, the Earth, and the Moon. The disturbances that concern us are not the absolute disturbance of each, but the relative disturbances of the Sun and Moon as viewed at the Earth. These will be rightly estimated by applying, to the Sun and the Moon, the disturbance of the Earth with changed sign, in addition to all disturbances to which they are liable from other causes, and thus, in fact, supposing the Earth to be stationary.

The Earth being stationary, the Sun goes round once in a year, and (omitting small periodical equations) with uniform angular velocity. At first he goes round in the original approximate plane of the Ecliptic, but in successive years he moves successively in different planes, all crossing the original plane in one line which passes through the Sun at a special point of his orbit, and more and more increasing their inclination to the original plane.

In the following investigation we shall neglect all perturbations of elements and coefficients of small inequalities, except that which is under our present special treatment.

Let $S = st$ be the Sun's longitude as seen from the Earth, s being constant.

$\sigma = \alpha\,t$, the Sun's linear elevation above the original plane.

R, the Sun's constant distance from the Earth.

$M = mt$, the Moon's longitude as seen from the Earth, m being constant.

r, the Moon's constant distance from the Earth.

z, the Moon's linear elevation at the time t, above the original plane.

The Sun's angular elevation as viewed from the Earth $= \sigma$

The Sun's disturbed angular elevation as viewed from the Moon

$$= \frac{\sigma}{R - r\,\cos\overline{M - S}} - \frac{z}{R - r\,\cos\overline{M - S}}$$

or sensibly $\frac{\sigma}{R} + \frac{\sigma\,r\,\cos\overline{M - S}}{R^2} - \frac{z}{R}$

The excess of the latter $= \frac{\sigma \cdot r \cdot \cos \overline{M - S}}{R^2} - \frac{z}{R}$

Producing the force which tends to elevate the Moon (omitting small terms)—

$$\frac{r \cdot \text{Sun's Mass}}{R^4} \cdot \sigma \cdot \cos \overline{M - S} - \text{Sun's Mass} \cdot \frac{z}{R^3}$$

The total force acting on the Moon to increase z is

$$- z \cdot \frac{\text{Earth's Mass}}{r^3} + \sigma \cdot \frac{r \cdot \text{Sun's Mass}}{R^4} \cdot \cos \overline{M - S} - z \cdot \frac{\text{Sun's Mass}}{R^3}$$

If the Sun's elevation increase uniformly, $\sigma = \alpha \cdot t$, where α is constant, and the total normal elevating force acting on the Moon $=$

$$z \times \left\{ - \frac{\text{Earth's Mass}}{r^3} - \frac{\text{Sun's Mass}}{R^3} \right\} + \frac{\alpha \cdot \text{Sun's Mass} \cdot r}{R^4} \cdot t \cdot \cos \overline{M - S},$$

where it will be remarked that the factor of z is the same as in an orbit where the change of Sun's elevation is not recognised

This force is the equivalent of $\frac{d^2 z}{dt^2}$

Substituting (for convenience) $- g^2$ for $\left\{ - \frac{\text{Earth's Mass}}{r^3} - \frac{\text{Sun's Mass}}{R^3} \right\}$, and

$+ k$ for $\frac{\alpha \cdot \text{Sun's Mass}}{R^4}$,

and putting vt for $\overline{M - S}$ or $\overline{mt - st}$,

$$\frac{d^2 z}{dt^2} + g^2 z = k \cdot t \cdot \cos vt$$

The general integral of the equation $\frac{d^2 z}{dt^2} + g^2 z = V$ is,—

$$z = \cos gt \int_t \frac{1}{\cos^2 gt} \int_t \cos gt \cdot V + A \cdot \cos gt + B \cdot \sin gt,$$

A and B being arbitraries (the same which occur in the solution for the equation in which the change of plane of orbit is not recognised)

It will be found here that $z = - \frac{k}{(m - s)^2 - g^2} \cdot t \cdot \cos \overline{M - S}$

$$+ \frac{2k(m - s)}{\{(m - s)^2 - g^2\}^2} \cdot \sin \overline{M - S} + A \cdot \cos gt + B \cdot \sin gt$$

The second term, whose only variable is $M - S$, represents that part of z which denotes a plane orbit in unvaried position, the first term, in which a nearly similar part is multiplied by a co-efficient of t, denotes an orbit uniformly changing its inclination to the original plane, with constant line of intersection

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NOTE

On the reference to the term $\frac{Zg - Z\sigma}{R}$ in Section IV, Part I, page 57

In alluding, page 53, to the introduction of a term $\frac{Zg - Z\sigma}{R}$, it is expressly stated that this is done, in order to take into account the change in the position of the Sun produced by the action of external planets It is however impossible, in this treatise, to enter into the details of solar or planetary perturbation as produced by these external causes, and therefore no distinct meaning can here be given to the terms $Z\sigma$, or Zg, as depending on that perturbing cause, or $\frac{Zg - Z\sigma}{R}$

NUMERICAL LUNAR THEORY

SECTION XIII.—INVESTIGATION OF THE EFFECT PRODUCED ON THE MOON'S MOTION BY GRADUAL CHANGE OF THE ELLIPTICITY OF THE EARTH'S ORBIT; ACCELERATION OF THE MOON'S MEAN MOTION PRODUCED BY DIMINUTION OF THE ELLIPTICITY.

The apparent geocentric movements of the Moon, as affected by the attraction of the Sun, are treated in the preceding Sections, on the supposition that the movements of the Center of Gravity of the System of "Earth and Moon" round the Sun are represented, very approximately, by the usual formulæ for elliptic motion. With these are to be combined elliptic movements of the Moon round the earth, and periodical disturbances (produced by the Sun) of the Moon's Geocentric Orbital Elements, producing a complicated effect on the Moon's geocentric position, but with no elementary change of a constantly progressive character. It has been thought, however, that researches in the planetary theory, have shown that some of the elements of heliocentric position of "Earth and Moon" undergo changes, which are small in amount, and in some measure periodical, but which, on the whole, produce in the Moon's geocentric place a progressive change, sensibly uniform, and continually in the same direction through exceedingly long periods.

It is the object of the present Section to examine the effect thus produced on the Moon's movements by the gradual diminution of the ellipticity of the orbit of "Earth and Moon" round the Sun.

The following notation will be employed throughout this section. It will be observed that the Earth is considered as the center of co-ordinates. It is understood that all motions are in the plane of the Ecliptic.

A, the mean distance R, the true distance, at time of observation	of the Sun
a, the mean or introductory distance r, the true distance, at time of observation	of the Moon
Nt, the mean longitude V, the true longitude, at time of observation	of the Sun
nt, the mean or introductory longitude v, the true longitude, at time of observation	of the Moon
E, the eccentricity, at time of observation S, the mean anomaly at time of observation	of the Solar Orbit
$\mathcal{R}$, the Sun's radial disturbing force on the Moon in direction from the Earth, at time of observation T, the Sun's tangential force on the Moon, at time of observation	produced by the Sun's attraction

The symbol Δ, connected in the first instance with E, and so forming ΔE, will in that combination denote the Variation of excentricity of the Solar Orbit at the time of observation, in other combinations, it will indicate the effect of that Variation of excentricity on other elements The first power only of Δ, and, in correspondence with it, the first power only of t, will be employed All the Variations are supposed to commence at the same common origin of time

The investigation, if attempted in its utmost generality, would be troublesome, principally from the great number of terms in its expansions To diminish this difficulty, we shall take the following measures As the object immediately sought is, the effect which is produced on the Moon's longitude, we shall entirely omit the consideration of the Moon's latitude, in so far as it can produce periodical terms And as we do not desire to investigate the excessively small changes (depending on the short periods which occur in lunar theories) connected with changes of the Moon's anomaly, &c, we shall entirely omit, from the expression for $\left(\frac{r}{a}\right)^2$, the periodical terms produced by excentricity or perturbations of the lunar orbit, preserving only the constant or non periodic term given in the first line of page 25, column 6 The arguments S, $(v-V)$, and their combinations, in the expansions shortly to be exhibited, are essential to the principal terms of the formulæ representing the Sun's action on the Moon

Subjected to the omissions above mentioned, the formulæ for the Sun's forces on the Moon become the following (*see* pages 60 and 61),

$$\left.\begin{array}{l} R = \text{Radial force} \\ \quad\text{on the Moon} \\ \quad\text{directed from} \\ \quad\text{the Earth} \end{array}\right\} = \left\{\begin{array}{l} +\ 00839 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right) \times \left\{\frac{1}{3} + \cos \overline{\left|2v-2V\right|}\right\} \\ + \left(\frac{A}{R}\right)^4 \left(\frac{r}{a}\right)^3 \times \left\{-\ 00002\ \cos \overline{\left|v-V\right|} -\ 00003\ \cos \overline{\left|3v-3V\right|}\right\} \end{array}\right\}$$

$$\left.\begin{array}{l} T = \text{Force ac-} \\ \quad\text{celerating the} \\ \quad\text{Moon's motion} \\ \quad\text{in longitude} \end{array}\right\} = \left\{\begin{array}{l} -\ 00839 \times \left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right) \times \sin \overline{\left|2v-2V\right|} \\ + \left(\frac{A}{R}\right)^4 \left(\frac{r}{a}\right)^3 \times \left\{+\ 00001\ \sin \overline{\left|v-V\right|} +\ 00003\ \sin \overline{\left|3v-3V\right|}\right\} \end{array}\right\}$$

the last figure in each multiplier being in the 5th decimal of semiaxis of the Earth's orbit round the Sun, and the unit of time being the mean of the times in which the Moon describes the angle 1, or $\frac{1}{84}$ of the Julian year, very nearly

In applying these formulæ, we shall omit the small terms multiplying $\overline{\left|v-V\right|}$ and $\overline{\left|3v-3V\right|}$, as not likely to produce results of sensible magnitude, and then multiplier $\left(\frac{A}{R}\right)^4$ will not be required

For $\left(\frac{A}{R}\right)^3$, the complete expansion given below will be used For the mean value of $\left(\frac{r}{a}\right)^2$, we shall use the constant value $+1\ 00469$ the first line in the table of Section II, Column 6, omitting entirely the periodic terms For $\overline{\left|2v-2V\right|}$ we shall use the complete expansions to be given below,

I We proceed now to give numerical values to the formulæ for the forces lately found, and we take, in the first place, the factor 00839, which applies to the two most important terms The number 00839 arises from the proportion of the Moon's mean distance a to the Sun's mean distance A, and considering the Sun's mean distance to be invariable, the number 00839 may be stated as "Parameter $\times\, a$", and its Variation will therefore be "Parameter $\times \Delta a$" Giving the proper value to the Parameter, (as derived from the present state of movements, and never changing so much as to disturb sensibly the proportion of Variations),

$$\Delta(00839) = \frac{00839}{a} \times \Delta a = 00839 \times \frac{\Delta a}{a}$$

The value of $\frac{\Delta a}{a}$ in terms of $\frac{\Delta E}{E}$ will be a matter of subsequent inquiry

II We must now form the numerical value of $\left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2$, with its Variation depending on ΔE and Δa Le Verrier's "expression" for $\frac{R}{A}$ is—

$$(+1\ 00000 + \overset{E}{00014}) - \overset{E}{01677} \cos S - \overset{E}{00014} \cos 2S,$$

where S is the Sun's mean anomaly, and the symbol's E^n and E placed above the co efficients denote the order of the powers of E which enter into those co efficients The number (the first in the line above), which is not attached to an argument, consists in fact of two parts, of which the second only depends on E Taking the reciprocal of this "expression," and forming its third power, the expanded form for $\left(\frac{A}{R}\right)^3$ is found to be

$$(+1\ 00001 - \overset{E^2}{00084}) + \overset{E}{05028} \cos S + \overset{E}{00252} \cos 2S$$

When multiplied by $+1\ 00469$ (the mean value of $\left(\frac{r}{a}\right)^2$) this becomes, for $\left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2$,

$$(1\ 00000 + \overset{\left(\frac{r}{a}\right)^2}{00470} - \overset{E}{00084}) + \overset{E}{05051} \cos S + \overset{E^2}{00253} \cos 2S$$

Its Variation is—

$$\left.\begin{array}{r} -\ 00940 \frac{\Delta a}{a} - 00168 \frac{\Delta E}{E} + 05051 \cos S \frac{\Delta E}{E} + 00506 \cos 2S \frac{\Delta E}{E} \\ -\ 0505 \sin S\ \Delta S - 00506 \sin 2S\ \Delta S \end{array}\right\}$$

Now S, the Sun's mean anomaly, depends only on the time elapsed, and is not in any way affected by the value of eccentricity, and, therefore, as connected with ΔE, $\Delta S = 0$ And the Variation of $\left(\frac{A}{R}\right)^3 \left(\frac{r}{a}\right)^2$ is reduced to

$$-\ 00940 \frac{\Delta a}{a} + \{-\ 00168 + 05051 \cos S + 00506 \cos 2S\} \times \frac{\Delta E}{E}$$

III The next term for expansion and Variation is cos $\overline{| 2v - 2V |}$ The Variation of this formulæ is $-\sin \overline{| 2v - 2V |} \times \Delta \overline{| 2v - 2V |}$ Now v (which is the Moon's longitude measured from the Moon's position when $t = 0$), may be expressed nearly by $Ct\ a^{-\frac{3}{2}}$, and ΔV will be $-\frac{3}{2}\ Ct\ a^{-\frac{5}{2}}\ \Delta a$ Therefore the first part of $\Delta \overline{| 2v - 2V |}$ is $-3\frac{v}{a}\ \Delta a = -3v\ \frac{\Delta a}{a}$

The second part of $\Delta \overline{| 2v - 2V |}$ is $-2\Delta V$ Here, $V = Nt +$ Solar Equation of the Center, $= Nt + 2E \sin S$, and $-2\Delta V = -4 \sin S\ \Delta E$ And the entire Variation of cos $\overline{| 2v - 2V |}$ is

$$-\sin \overline{| 2v - 2V |} \times \left\{ -3v\ \frac{\Delta a}{a} - 4 \sin S\ \Delta E \right\}$$

IV The fourth term gives a result for Variation which differs from that of the third term in its trigonometrical part, only in adopting the multiplier $+ \cos \overline{| 2v - 2V |}$ instead of $- \sin \overline{| 2v - 2V |}$ It gives for Variation of sin $\overline{| 2v - 2V |}$,

$$+ \cos \overline{| 2v - 2V |} \times \left\{ -3v\ \frac{\Delta a}{a} - 4 \sin S\ \Delta E \right\}$$

We shall now substitute, in the combinations for forming R and T, the values which we have lately exhibited, and shall multiply together the separate lines as consecutive factors As has been stated before, we proceed only to the first order of ΔE or Δa By arranging the subordinate parts of each value in two groups, principal terms and small terms, we shall have occasion only to multiply each group of small terms in one value, by the products of the principal terms of the other values, and take the sum of the products

To form the Variation of R, the radial force on the Moon —

FIRST PART

First factor, $\frac{+\ 00839}{3} \quad = \quad +\ 00240 \ +\ 00240 \times \frac{\Delta a}{a}$

Second factor, $\left(\frac{A}{R}\right)^3 \quad = \quad +1\ 00001 + (-\ 00168 + 05028 \cos S + 00504 \cos 2S) \times \frac{\Delta E}{E}$

Third factor, constant for $\left(\frac{1}{a}\right)^2 = +1\ 00469 \ +\ 00940\ \frac{\Delta a}{a}$

The multiplications above mentioned will be the following, the terms which are obviously periodical being omitted,

$$+1\ 00001 \times +1\ 00469 \times +\ 00240 \times \frac{\Delta a}{a} = +\ 00241 \times \frac{\Delta a}{a}$$

$$+0\ 00240 \times +1\ 00469 \times -\ 00168 \times \frac{\Delta E}{E} = -\ 00000 \times \frac{\Delta E}{E}$$

$$+0\ 00240 \times +1\ 00001 \times -\ 00940 \times \frac{\Delta a}{a} = -\ 00002 \times \frac{\Delta a}{a}$$

SECOND PART

Fiist factoi, $+\ 00839 \quad = \quad +\ 00240 \quad +\ 00240 \times \frac{\Delta a}{a}$

Second factor, $\left(\frac{A}{R}\right)^2 \quad = \quad +\ 1\ 00001 \quad + (+\ 00168 + 05028\ \cos S$

$$+\ 00504\ \cos 2S) \times \frac{\Delta E}{E}$$

Thiid factoi, constant for $\left(\frac{r}{a}\right)^2 \quad = +\ 1\ \ 00469$

Fourth factor, $\cos\ \overline{2v - 2V} \quad = +\cos\ \overline{2v - 2V}$

$$-\sin\ \overline{2v - 2V} \times \left(-3v\ \frac{\Delta a}{a} - 4 \sin S\ \ \Delta E\right)$$

The results of the multiplications are all periodical

It appears therefore that the only non-periodical pait in the radial force on the Moon is $+\ 00240 \times \frac{\Delta a}{a}$

Foi the Vaiiation of T, the tangential force on the Moon, on iemaiking that its expiession, expanded in the same order, will contain no term corresponding to the "First Factoi" of the "Second Pait" above, and that, in other paits, sine and cosine are precisely interchanged, it will be seen at once that the Variation of T has no non-periodical term

It appeais therefore that, as far as the first power of small quantities, no change is produced in the Moon's mean longitude by a change in the excentricity of the Earth's Orbit round the Sun

G B AIRY

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